

Functions and Graphs

- 1.1** Modeling and Equation Solving
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One of the central principles of economics is that the value of money is not constant; it is a function of time. Since fortunes are made and lost by people attempting to predict the future value of money, much attention is paid to quantitative measures like the consumer price index, a basic measure of inflation in various sectors of the economy. See page 159 for a look at how the consumer price index for housing has behaved over time.

Chapter 1 Overview

In this chapter we begin the study of functions that will continue throughout the book. Your previous courses have introduced you to some basic functions. These functions can be visualized using a graphing calculator, and their properties can be described using the notation and terminology that will be introduced in this chapter. A familiarity with this terminology will serve you well in later chapters when we explore properties of functions in greater depth.

1.1

Modeling and Equation Solving

What you'll learn about

- Numerical Models
- Algebraic Models
- Graphical Models
- The Zero Factor Property
- Problem Solving
- Grapher Failure and Hidden Behavior
- A Word About Proof

... and why

Numerical, algebraic, and graphical models provide different methods to visualize, analyze, and understand data.

Numerical Models

Scientists and engineers have always used mathematics to model the real world and thereby to unravel its mysteries. A **mathematical model** is a mathematical structure that approximates phenomena for the purpose of studying or predicting their behavior. Thanks to advances in computer technology, the process of devising mathematical models is now a rich field of study itself, **mathematical modeling**.

We will be concerned primarily with three types of mathematical models in this book: *numerical models*, *algebraic models*, and *graphical models*. Each type of model gives insight into real-world problems, but the best insights are often gained by switching from one kind of model to another. Developing the ability to do that will be one of the goals of this course.

Perhaps the most basic kind of mathematical model is the **numerical model**, in which numbers (or *data*) are analyzed to gain insights into phenomena. A numerical model can be as simple as the major league baseball standings or as complicated as the network of interrelated numbers that measure the global economy.

EXAMPLE 1 Tracking the Minimum Wage

The numbers in Table 1.1 show the growth of the minimum hourly wage (MHW) from 1955 to 2005. The table also shows the MHW adjusted to the purchasing power of 1996 dollars (using the CPI-U, the Consumer Price Index for all Urban Consumers). Answer the following questions using only the data in the table.

- (a) In what five-year period did the actual MHW increase the most?
- (b) In what year did a worker earning the MHW enjoy the greatest purchasing power?
- (c) A worker on minimum wage in 1980 was earning nearly twice as much as a worker on minimum wage in 1970, and yet there was great pressure to raise the minimum wage again. Why?

SOLUTION

- (a) In the period 1975 to 1980 it increased by \$1.00. Notice that the minimum wage never goes down, so we can tell that there were no other increases of this magnitude even though the table does not give data from every year.
- (b) In 1970.
- (c) Although the MHW increased from \$1.60 to \$3.10 in that period, the purchasing power actually dropped by \$0.57 (in 1996 dollars). This is one way inflation can affect the economy.

Now try Exercise 11.



Table 1.1 The Minimum Hourly Wage

Year	Minimum Hourly Wage (MHW)	Purchasing Power in 1996 Dollars
1955	0.75	4.39
1960	1.00	5.30
1965	1.25	6.23
1970	1.60	6.47
1975	2.10	6.12
1980	3.10	5.90
1985	3.35	4.88
1990	3.80	4.56
1995	4.25	4.38
2000	5.15	4.69
2005	5.15	4.15

Source: www.infoplease.com



Table 1.2 U.S. Prison Population (Thousands)

Year	Total	Male	Female
1980	316	304	12
1985	480	459	21
1990	740	699	41
1995	1085	1021	64
2000	1382	1290	92

Source: U.S. Justice Department.

Table 1.3 Female Percentage of U.S. Prison Population

Year	% Female
1980	3.8
1985	4.4
1990	5.5
1995	5.9
2000	6.7

Source: U.S. Justice Department.

The numbers in Table 1.1 provide a numerical model for one aspect of the U.S. economy by using another numerical model, the urban consumer price index (CPI-U), to adjust the data. Working with large numerical models is standard operating procedure in business and industry, where computers are relied upon to provide fast and accurate data processing.

EXAMPLE 2 Analyzing Prison Populations

Table 1.2 shows the growth in the number of prisoners incarcerated in state and federal prisons from 1980 to 2000. Is the proportion of female prisoners over the years increasing?

SOLUTION The *number* of female prisoners over the years is certainly increasing, but so is the total number of prisoners, so it is difficult to discern from the data whether the *proportion* of female prisoners is increasing. What we need is another column of numbers showing the ratio of female prisoners to total prisoners.

We could compute all the ratios separately, but it is easier to do this kind of repetitive calculation with a single command on a computer spreadsheet. You can also do this on a graphing calculator by manipulating lists (see Exercise 19). Table 1.3 shows the percentage of the total population each year that consists of female prisoners. With this data to extend our numerical model, it is clear that the proportion of female prisoners is increasing.

Now try Exercise 19.

Algebraic Models

An **algebraic model** uses formulas to relate variable quantities associated with the phenomena being studied. The added power of an algebraic model over a numerical model is that it can be used to generate numerical values of unknown quantities by relating them to known quantities.

EXAMPLE 3 Comparing Pizzas

A pizzeria sells a rectangular 18" by 24" pizza for the same price as its large round pizza (24" diameter). If both pizzas are of the same thickness, which option gives the most pizza for the money?

SOLUTION We need to compare the *areas* of the pizzas. Fortunately, geometry has provided algebraic models that allow us to compute the areas from the given information.

For the rectangular pizza:

$$\text{Area} = l \times w = 18 \times 24 = 432 \text{ square inches.}$$

For the circular pizza:

$$\text{Area} = \pi r^2 = \pi \left(\frac{24}{2} \right)^2 = 144\pi \approx 452.4 \text{ square inches.}$$

The round pizza is larger and therefore gives more for the money.

Now try Exercise 21.

The algebraic models in Example 3 come from geometry, but you have probably encountered algebraic models from many other sources in your algebra and science courses.

EXPLORATION EXTENSIONS

Suppose that after the sale, the merchandise prices are increased by 25%. If m represents the marked price before the sale, find an algebraic model for the post-sale price, including tax.

EXPLORATION 1
Designing an Algebraic Model

A department store is having a sale in which everything is discounted 25% off the marked price. The discount is taken at the sales counter, and then a state sales tax of 6.5% and a local sales tax of 0.5% are added on.

1. The discount price d is related to the marked price m by the formula $d = km$, where k is a certain constant. What is k ?
2. The actual sale price s is related to the discount price d by the formula $s = d + td$, where t is a constant related to the total sales tax. What is t ?
3. Using the answers from steps 1 and 2 you can find a constant p that relates s directly to m by the formula $s = pm$. What is p ?
4. If you only have \$30, can you afford to buy a shirt marked \$36.99?
5. If you have a credit card but are determined to spend no more than \$100, what is the maximum total value of your marked purchases before you present them at the sales counter?

The ability to generate numbers from formulas makes an algebraic model far more useful as a predictor of behavior than a numerical model. Indeed, one optimistic goal of scientists and mathematicians when modeling phenomena is to fit an algebraic model to numerical data and then (even more optimistically) to analyze why it works. Not all models can be used to make accurate predictions. For example, nobody has ever devised a successful formula for predicting the ups and downs of the stock market as a function of time, although that does not stop investors from trying.

If numerical data do behave reasonably enough to suggest that an algebraic model might be found, it is often helpful to look at a picture first. That brings us to graphical models.

Graphical Models

A **graphical model** is a visible representation of a numerical model or an algebraic model that gives insight into the relationships between variable quantities. Learning to interpret and use graphs is a major goal of this book.

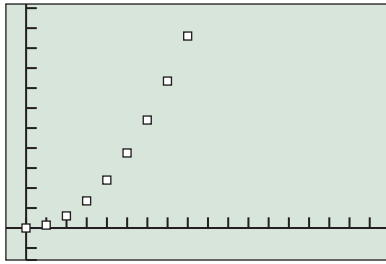
EXAMPLE 4
Visualizing Galileo's Gravity Experiments

Galileo Galilei (1564–1642) spent a good deal of time rolling balls down inclined planes carefully recording the distance they traveled as a function of elapsed time. His experiments are commonly repeated in physics classes today, so it is easy to reproduce a typical table of Galilean data.

Elapsed time (seconds)	0	1	2	3	4	5	6	7	8
Distance traveled (inches)	0	0.75	3	6.75	12	18.75	27	36.75	48

What graphical model fits the data? Can you find an algebraic model that fits?

continued



$[-1, 18]$ by $[-8, 56]$

FIGURE 1.1 A scatter plot of the data from a Galileo gravity experiment. (Example 4)

SOLUTION A scatter plot of the data is shown in Figure 1.1.

Galileo's experience with quadratic functions suggested to him that this figure was a parabola with its vertex at the origin; he therefore modeled the effect of gravity as a quadratic function:

$$d = kt^2.$$

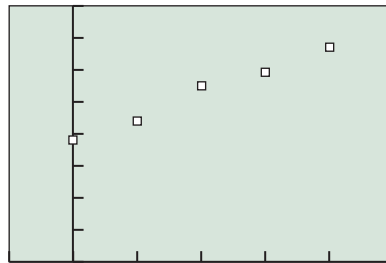
Because the ordered pair $(1, 0.75)$ must satisfy the equation, it follows that $k = 0.75$, yielding the equation

$$d = 0.75t^2.$$

You can verify numerically that this algebraic model correctly predicts the rest of the data points. We will have much more to say about parabolas in Chapter 2.

Now try Exercise 23.

This insight led Galileo to discover several basic laws of motion that would eventually be named after Isaac Newton. While Galileo had found the algebraic model to describe the path of the ball, it would take Newton's calculus to explain why it worked.



$[-5, 25]$ by $[0, 8]$

FIGURE 1.2 A scatter plot of the data in Table 1.4. (Example 5)

EXAMPLE 5 Fitting a Curve to Data

We showed in Example 2 that the percentage of females in the U.S. prison population has been steadily growing over the years. Model this growth graphically and use the graphical model to suggest an algebraic model.

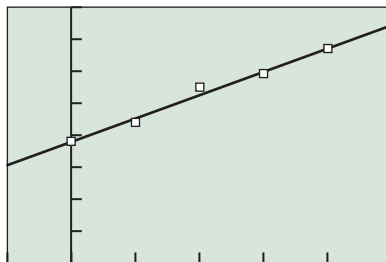
SOLUTION Let t be the number of years after 1980, and let F be the percentage of females in the prison population from year 0 to year 20. From the data in Table 1.3 we get the corresponding data in Table 1.4:



Table 1.4 Percentage (F) of Females in the Prison Population t years after 1980

t	0	5	10	15	20
F	3.8	4.4	5.5	5.9	6.7

Source: U.S. Justice Department.



$[-5, 25]$ by $[0, 8]$

FIGURE 1.3 The line with equation $y = 0.145x + 3.8$ is a good model for the data in Table 1.4. (Example 5)

A scatter plot of the data is shown in Figure 1.2.

This pattern looks linear. If we use a line as our graphical model, we can find an algebraic model by finding the equation of the line. We will describe in Chapter 2 how a statistician would find the best line to fit the data, but we can get a pretty good fit for now by finding the line through the points $(0, 3.8)$ and $(20, 6.7)$.

The slope is $(6.7 - 3.8)/(20 - 0) = 0.145$ and the y -intercept is 3.8. Therefore, the line has equation $y = 0.145x + 3.8$. You can see from Figure 1.3 that this line does a very nice job of modeling the data.

Now try Exercises 13 and 15.

EXPLORATION EXTENSIONS

What are the advantages of a linear model over a quadratic model for these data?

EXPLORATION 2 Interpreting the Model

The parabola in Example 4 arose from a law of physics that governs falling objects, which should inspire more confidence than the linear model in Example 5. We can repeat Galileo's experiment many times with differently-sloped ramps, different units of measurement, and even on different planets, and a quadratic model will fit it every time. The purpose of this Exploration is to think more deeply about the linear model in the prison example.

1. The linear model we found will not continue to predict the percentage of female prisoners in the U.S. indefinitely. Why must it eventually fail?
2. Do you think that our linear model will give an accurate estimate of the percentage of female prisoners in the U.S. in 2009? Why or why not?
3. The linear model is such a good fit that it actually calls our attention to the unusual jump in the percentage of female prisoners in 1990. Statisticians would look for some unusual "confounding" factor in 1990 that might explain the jump. What sort of factors do you think might explain it?
4. Does Table 1.1 suggest a possible factor that might influence female crime statistics?

PREREQUISITE CHAPTER

In the Prerequisite Chapter we defined solution of an equation, solving an equation, x-intercept, and graph of an equation in x and y .

There are other ways of graphing numerical data that are particularly useful for statistical studies. We will treat some of them in Chapter 9. The scatter plot will be our choice of data graph for the time being, as it provides the closest connection to graphs of functions in the Cartesian plane.

The Zero Factor Property

The main reason for studying algebra through the ages has been to solve equations. We develop algebraic models for phenomena so that we can solve problems, and the solutions to the problems usually come down to finding solutions of algebraic equations.

If we are fortunate enough to be solving an equation in a single variable, we might proceed as in the following example.

**EXAMPLE 6 Solving an Equation Algebraically**

Find all real numbers x for which $6x^3 = 11x^2 + 10x$.

SOLUTION We begin by changing the form of the equation to $6x^3 - 11x^2 - 10x = 0$.

We can then solve this equation algebraically by factoring:

$$6x^3 - 11x^2 - 10x = 0$$

$$x(6x^2 - 11x - 10) = 0$$

$$x(2x - 5)(3x + 2) = 0$$

continued

$$x = 0 \quad \text{or} \quad 2x - 5 = 0 \quad \text{or} \quad 3x + 2 = 0$$

$$x = 0 \quad \text{or} \quad x = \frac{5}{2} \quad \text{or} \quad x = -\frac{2}{3}$$

Now try Exercise 31.

In Example 6, we used the important Zero Factor Property of real numbers.

The Zero Factor Property

A product of real numbers is zero if and only if at least one of the factors in the product is zero.

It is this property that algebra students use to solve equations in which an expression is set equal to zero. Modern problem solvers are fortunate to have an alternative way to find such solutions.

If we graph the expression, then the x -intercepts of the graph of the expression will be the values for which the expression equals 0.

EXAMPLE 7 Solving an Equation: Comparing Methods

Solve the equation $x^2 = 10 - 4x$.

SOLUTION

Solve Algebraically

The given equation is equivalent to $x^2 + 4x - 10 = 0$.

This quadratic equation has irrational solutions that can be found by the quadratic formula.

$$x = \frac{-4 + \sqrt{16 + 40}}{2} \approx 1.7416574$$

and

$$x = \frac{-4 - \sqrt{16 + 40}}{2} \approx -5.7416574$$

While the decimal answers are certainly accurate enough for all practical purposes, it is important to note that only the expressions found by the quadratic formula give the *exact* real number answers. The tidiness of exact answers is a worthy mathematical goal. Realistically, however, exact answers are often impossible to obtain, even with the most sophisticated mathematical tools.

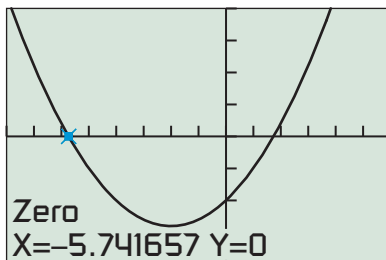
Solve Graphically

We first find an equivalent equation with 0 on the right-hand side: $x^2 + 4x - 10 = 0$. We then graph the equation $y = x^2 + 4x - 10$, as shown in Figure 1.4.

We then use the grapher to locate the x -intercepts of the graph:

$$x \approx 1.7416574 \quad \text{and} \quad x \approx -5.741657.$$

Now try Exercise 35.



$[-8, 6]$ by $[-20, 20]$

FIGURE 1.4 The graph of $y = x^2 + 4x - 10$. (Example 7)

SOLVING EQUATIONS WITH TECHNOLOGY

Example 7 shows one method of solving an equation with technology. Some graphers could also solve the equation in Example 7 by finding the *intersection* of the graphs of $y = x^2$ and $y = 10 - 4x$. Some graphers have built-in equation solvers. Each method has its advantages and disadvantages, but we recommend the “finding the x -intercepts” technique for now because it most closely parallels the classical algebraic techniques for finding roots of equations, and makes the connection between the algebraic and graphical models easier to follow and appreciate.

We used the graphing utility of the calculator to **solve graphically** in Example 7. Most calculators also have solvers that would enable us to **solve numerically** for the same decimal approximations without considering the graph. Some calculators have computer algebra systems that will solve numerically to produce exact answers in certain cases. In this book we will distinguish between these two technological methods and the traditional pencil-and-paper methods used to **solve algebraically**.

Every method of solving an equation usually comes down to finding where an expression equals zero. If we use $f(x)$ to denote an algebraic expression in the variable x , the connections are as follows:

Fundamental Connection

If a is a real number that solves the equation $f(x) = 0$, then these three statements are equivalent:

1. The number a is a **root** (or **solution**) of the **equation** $f(x) = 0$.
2. The number a is a **zero** of $y = f(x)$.
3. The number a is an **x -intercept** of the **graph** of $y = f(x)$. (Sometimes the point $(a, 0)$ is referred to as an x -intercept.)

Problem Solving

George Pólya (1887–1985) is sometimes called the father of modern problem solving, not only because he was good at it (as he certainly was) but also because he published the most famous analysis of the problem-solving process: *How to Solve It: A New Aspect of Mathematical Method*. His “four steps” are well known to most mathematicians:

Pólya's Four Problem-Solving Steps

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Look back.

The problem-solving process that we recommend you use throughout this course will be the following version of Pólya's four steps.

A Problem-Solving Process

Step 1—Understand the problem.

- Read the problem as stated, several times if necessary.
- Be sure you understand the meaning of each term used.
- Restate the problem in your own words. Discuss the problem with others if you can.
- Identify clearly the information that you need to solve the problem.
- Find the information you need from the given data.

Step 2—Develop a mathematical model of the problem.

- Draw a picture to visualize the problem situation. It usually helps.
- Introduce a variable to represent the quantity you seek. (In some cases there may be more than one.)
- Use the statement of the problem to find an equation or inequality that relates the variables you seek to quantities that you know.

Step 3—Solve the mathematical model and support or confirm the solution.

- **Solve algebraically** using traditional algebraic methods and **support graphically or support numerically** using a graphing utility.
- **Solve graphically or numerically** using a graphing utility and **confirm algebraically** using traditional algebraic methods.
- **Solve graphically or numerically** because there is no other way possible.

Step 4—Interpret the solution in the problem setting.

- Translate your mathematical result into the problem setting and decide whether the result makes sense.

EXAMPLE 8 Applying the Problem-Solving Process

The engineers at an auto manufacturer pay students \$0.08 per mile plus \$25 per day to road test their new vehicles.

- (a) How much did the auto manufacturer pay Sally to drive 440 miles in one day?
 (b) John earned \$93 test-driving a new car in one day. How far did he drive?

SOLUTION

Model A picture of a car or of Sally or John would not be helpful, so we go directly to designing the model. Both John and Sally earned \$25 for one day, plus \$0.08 per mile. Multiply dollars/mile by miles to get dollars.

So if p represents the pay for driving x miles in one day, our algebraic model is

$$p = 25 + 0.08x.$$

Solve Algebraically

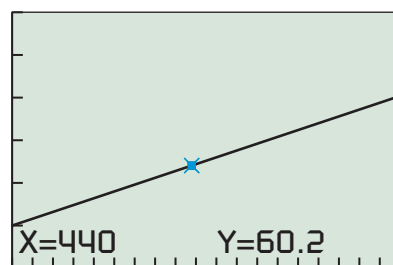
- (a) To get Sally's pay we let $x = 440$ and solve for p :

$$\begin{aligned} p &= 25 + 0.08(440) \\ &= 60.20 \end{aligned}$$

- (b) To get John's mileage we let $p = 93$ and solve for x :

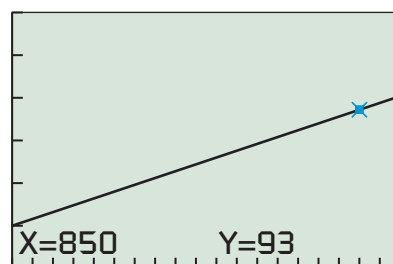
$$\begin{aligned} 93 &= 25 + 0.08x \\ 68 &= 0.08x \\ x &= \frac{68}{0.08} \\ x &= 850 \end{aligned}$$

continued



[0, 940] by [0, 150]

(a)



[0, 940] by [0, 150]

(b)

FIGURE 1.5 Graphical support for the algebraic solutions in Example 8.

TECHNOLOGY NOTE

One way to get the table in Figure 1.6b is to use the “Ask” feature of your graphing calculator and enter each x value separately.

Support Graphically

Figure 1.5a shows that the point $(440, 60.20)$ is on the graph of $y = 25 + 0.08x$, supporting our answer to (a). Figure 1.5b shows that the point $(850, 93)$ is on the graph of $y = 25 + 0.08x$, supporting our answer to (b). (We could also have **supported** our answer **numerically** by simply substituting in for each x and confirming the value of p .)

Interpret

Sally earned \$60.20 for driving 440 miles in one day. John drove 850 miles in one day to earn \$93.00.

Now try Exercise 47.

It is not really necessary to *show* written support as part of an algebraic solution, but it is good practice to support answers wherever possible simply to reduce the chance for error. We will often show written support of our solutions in this book in order to highlight the connections among the algebraic, graphical, and numerical models.

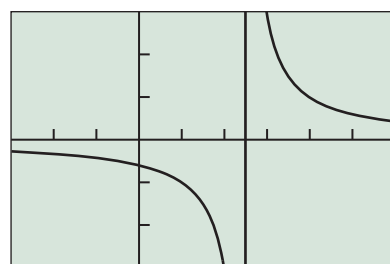
Grapher Failure and Hidden Behavior

While the graphs produced by computers and graphing calculators are wonderful tools for understanding algebraic models and their behavior, it is important to keep in mind that machines have limitations. Occasionally they can produce graphical models that misrepresent the phenomena we wish to study, a problem we call **grapher failure**. Sometimes the viewing window will be too large, obscuring details of the graph which we call **hidden behavior**. We will give an example of each just to illustrate what can happen, but rest assured that these difficulties rarely occur with graphical models that arise from real-world problems.

EXAMPLE 9 Seeing Grapher Failure

Look at the graph of $y = 3/(2x - 5)$ on a graphing calculator. Is there an x -intercept?

SOLUTION A graph is shown in Figure 1.6a.



[-3, 6] by [-3, 3]

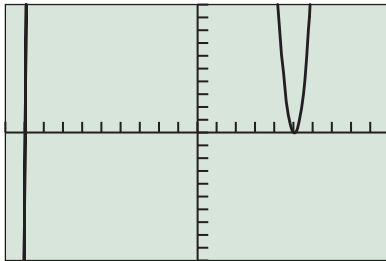
(a)

X	Y1	
2.4	-15	
2.49	-150	
2.499	-1500	
2.5	ERROR	
2.501	1500	
2.51	150	
2.6	15	
Y1 = 3/(2X-5)		

(b)

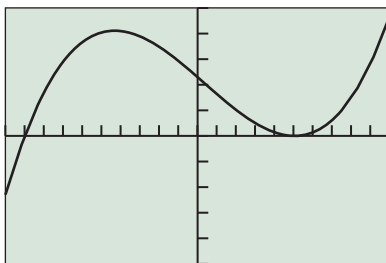
FIGURE 1.6 (a) A graph with a mysterious x -intercept. (b) As x approaches 2.5, the value of $3/(2x - 5)$ approaches $\pm\infty$. (Example 9)

continued



$[-10, 10]$ by $[-10, 10]$

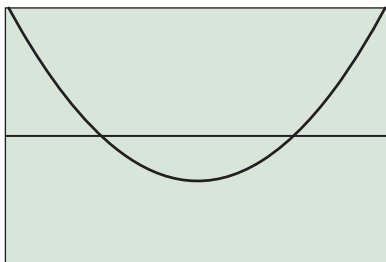
(a)



$[-10, 10]$ by $[-500, 500]$

(b)

FIGURE 1.7 The graph of $y = x^3 - 1.1x^2 - 65.4x + 229.5$ in two viewing windows. (Example 10)



$[4.95, 5.15]$ by $[-0.1, 0.1]$

FIGURE 1.8 A closer look at the graph of $y = x^3 - 1.1x^2 - 65.4x + 229.5$. (Example 10)

The graph seems to show an x -intercept about halfway between 2 and 3. To confirm this algebraically, we would set $y = 0$ and solve for x :

$$\begin{aligned} 0 &= \frac{3}{2x - 5} \\ 0(2x - 5) &= 3 \\ 0 &= 3 \end{aligned}$$

The statement $0 = 3$ is false for all x , so there can be no value that makes $y = 0$, and hence there can be no x -intercept for the graph. What went wrong?

The answer is a simple form of grapher failure: The vertical line should not be there! As suggested by the table in Figure 1.6b, the actual graph of $y = 3/(2x - 5)$ approaches $-\infty$ to the left of $x = 2.5$, and comes down from $+\infty$ to the right of $x = 2.5$ (more on this later). The expression $3/(2x - 5)$ is undefined at $x = 2.5$, but the graph in Figure 1.6a does not reflect this. The grapher plots points at regular increments from left to right, connecting the points as it goes. It hits some low point off the screen to the left of 2.5, followed immediately by some high point off the screen to the right of 2.5, and it connects them with that unwanted line.

Now try Exercise 49.

EXAMPLE 10 Not Seeing Hidden Behavior

Solve graphically: $x^3 - 1.1x^2 - 65.4x + 229.5 = 0$.

SOLUTION Figure 1.7a shows the graph in the standard $[-10, 10]$ by $[-10, 10]$ window, an inadequate choice because too much of the graph is off the screen. Our horizontal dimensions look fine, so we adjust our vertical dimensions to $[-500, 500]$, yielding the graph in Figure 1.7b.

We use the grapher to locate an x -intercept near -9 (which we find to be -9) and then an x -intercept near 5 (which we find to be 5). The graph leads us to believe that we are done. However, if we zoom in closer to observe the behavior near $x = 5$, the graph tells a new story (Figure 1.8).

In this graph we see that there are actually *two* x -intercepts near 5 (which we find to be 5 and 5.1). There are therefore three roots (or zeros) of the equation $x^3 - 1.1x^2 - 65.4x + 229.5 = 0$: $x = -9$, $x = 5$, and $x = 5.1$.

Now try Exercise 51.

You might wonder if there could be still *more* hidden x -intercepts in Example 10! We will learn in Chapter 2 how the *Fundamental Theorem of Algebra* guarantees that there are not.

A Word About Proof

While Example 10 is still fresh on our minds, let us point out a subtle, but very important, consideration about our solution.

We *solved graphically* to find two solutions, then eventually three solutions, to the given equation. Although we did not show the steps, it is easy to *confirm numerically* that the three numbers found are actually solutions by substituting them into the equation. But the problem asked us to find *all* solutions. While we could explore that equation

TEACHER NOTE

Sometimes it is impossible to show all of the details of a graph in a single window. For example, in Example 10 the graph in Figure 1.8 reveals minute details of the graph, but it hides the overall shape of the graph.

graphically in a hundred more viewing windows and never find another solution, our failure to find them would not *prove* that they are not out there somewhere. That is why the Fundamental Theorem of Algebra is so important. It tells us that there can be at most three real solutions to *any* cubic equation, so we know for a fact that there are no more.

Exploration is encouraged throughout this book because it is how mathematical progress is made. Mathematicians are never satisfied, however, until they have *proved* their results. We will show you proofs in later chapters and we will ask you to produce proofs occasionally in the exercises. That will be a time for you to set the technology aside, get out a pencil, and show in a logical sequence of algebraic steps that something is undeniably and universally true. This process is called **deductive reasoning**.

EXAMPLE 11 Proving a Peculiar Number Fact

Prove that 6 is a factor of $n^3 - n$ for every positive integer n .

SOLUTION You can explore this expression for various values of n on your calculator. Table 1.5 shows it for the first 12 values of n .

Table 1.5 The first 12 values of $n^3 - n$												
n	1	2	3	4	5	6	7	8	9	10	11	12
$n^3 - n$	0	6	24	60	120	210	336	504	720	990	1320	1716

All of these numbers are divisible by 6, but that does not prove that they will continue to be divisible by 6 for all values of n . In fact, a table with a billion values, all divisible by 6, would not constitute a proof. Here is a proof:

Let n be *any* positive integer.

- We can factor $n^3 - n$ as the product of three numbers:
 $(n - 1)(n)(n + 1)$.
- The factorization shows that $n^3 - n$ is always the product of three consecutive integers.
- Every set of three consecutive integers must contain a multiple of 3.
- Since 3 divides a factor of $n^3 - n$, it follows that 3 is a factor of $n^3 - n$ itself.
- Every set of three consecutive integers must contain a multiple of 2.
- Since 2 divides a factor of $n^3 - n$, it follows that 2 is a factor of $n^3 - n$ itself.
- Since both 2 and 3 are factors of $n^3 - n$, we know that 6 is a factor of $n^3 - n$.

End of proof!

Now try Exercise 53.

QUICK REVIEW 1.1 (For help, go to Section A.2.)

Factor the following expressions completely over the real numbers.

1. $x^2 - 16$

2. $x^2 + 10x + 25$

3. $81y^2 - 4$

4. $3x^3 - 15x^2 + 18x$

5. $16h^4 - 81$

7. $x^2 + 3x - 4$

9. $2x^2 - 11x + 5$

6. $x^2 + 2xh + h^2$

8. $x^2 - 3x + 4$

10. $x^4 + x^2 - 20$

SECTION 1.1 EXERCISES

In Exercises 1–10, match the numerical model to the corresponding graphical model (a–j) and algebraic model (k–t).

1.

x	3	5	7	9	12	15
y	6	10	14	18	24	30

2.

x	0	1	2	3	4	5
y	2	3	6	11	18	27

3.

x	2	4	6	8	10	12
y	4	10	16	22	28	34

4.

x	5	10	15	20	25	30
y	90	80	70	60	50	40

5.

x	1	2	3	4	5	6
y	39	36	31	24	15	4

6.

x	1	2	3	4	5	6
y	5	7	9	11	13	15

7.

x	5	7	9	11	13	15
y	1	2	3	4	5	6

8.

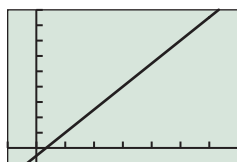
x	4	8	12	14	18	24
y	20	72	156	210	342	600

9.

x	3	4	5	6	7	8
y	8	15	24	35	48	63

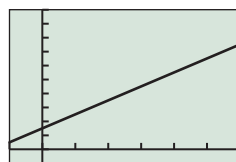
10.

x	4	7	12	19	28	39
y	1	2	3	4	5	6



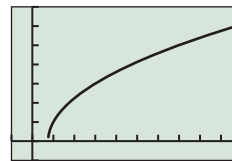
[-2, 14] by [-4, 36]

(a)



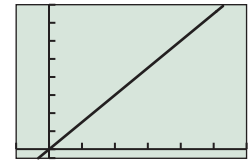
[-1, 6] by [-2, 20]

(b)



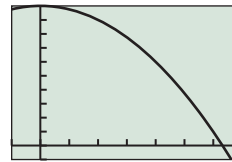
[-4, 40] by [-1, 7]

(c)



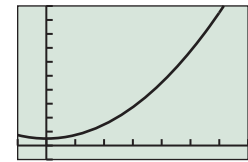
[-3, 18] by [-2, 32]

(d)



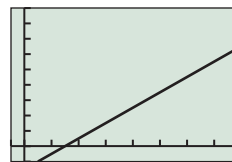
[-1, 7] by [-4, 40]

(e)



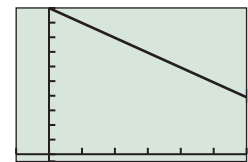
[-1, 7] by [-4, 40]

(f)



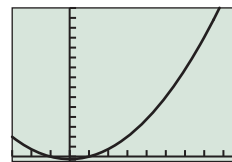
[-1, 16] by [-1, 9]

(g)



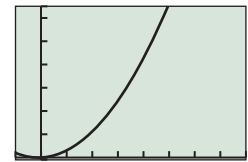
[-5, 30] by [-5, 100]

(h)



[-3, 9] by [-2, 60]

(i)



[-5, 40] by [-10, 650]

(j)

(k) $y = x^2 + x$

(l) $y = 40 - x^2$

(m) $y = (x + 1)(x - 1)$

(n) $y = \sqrt{x - 3}$

(o) $y = 100 - 2x$

(p) $y = 3x - 2$

(q) $y = 2x$

(r) $y = x^2 + 2$

(s) $y = 2x + 3$

(t) $y = \frac{x - 3}{2}$

Exercises 11–18 refer to the data in Table 1.6 below showing the percentage of the female and male populations in the United States employed in the civilian work force in selected years from 1954 to 2004.



Table 1.6 Employment Statistics

Year	Female	Male
1954	32.3	83.5
1959	35.1	82.3
1964	36.9	80.9
1969	41.1	81.1
1974	42.8	77.9
1979	47.7	76.5
1984	50.1	73.2
1989	54.9	74.5
1994	56.2	72.6
1999	58.5	74.0
2004	57.4	71.9

Source: www.bls.gov

11. (a) According to the numerical model, what has been the trend in females joining the work force since 1954?
(b) In what 5-year interval did the percentage of women who were employed change the most?
12. (a) According to the numerical model, what has been the trend in males joining the work force since 1954?
(b) In what 5-year interval did the percentage of men who were employed change the most?
13. Model the data graphically with two scatter plots on the same graph, one showing the percentage of women employed as a function of time and the other showing the same for men. Measure time in years since 1954.
14. Are the male percentages falling faster than the female percentages are rising, or vice versa?
15. Model the data algebraically with linear equations of the form $y = mx + b$. Write one equation for the women's data and another equation for the men's data. Use the 1954 and 1999 ordered pairs to compute the slopes.
16. If the percentages continue to follow the linear models you found in Exercise 15, what will the employment percentages for women and men be in the year 2009?
17. If the percentages continue to follow the linear models you found in Exercise 15, when will the percentages of women and men in the civilian work force be the same? What percentage will that be?
18. **Writing to Learn** Explain why the percentages cannot continue indefinitely to follow the linear models that you wrote in Exercise 15.
19. **Doing Arithmetic with Lists** Enter the data from the "Total" column of Table 1.2 of Example 2 into list L_1 in your calculator. Enter the data from the "Female" column into list L_2 . Check a few computations to see that the procedures in (a) and (b) cause the cal-

culator to divide each element of L_2 by the corresponding entry in L_1 , multiply it by 100, and store the resulting list of percentages in L_3 .

- (a) On the home screen, enter the command: $100 \times L_2 / L_1 \rightarrow L_3$.
- (b) Go to the top of list L_3 and enter $L_3 = 100(L_2/L_1)$.

20. **Comparing Cakes** A bakery sells a 9" by 13" cake for the same price as an 8" diameter round cake. If the round cake is twice the height of the rectangular cake, which option gives the most cake for the money?
21. **Stepping Stones** A garden shop sells 12" by 12" square stepping stones for the same price as 13" round stones. If all of the stepping stones are the same thickness, which option gives the most rock for the money?
22. **Free Fall of a Smoke Bomb** At the Oshkosh, WI, air show, Jake Troupers drops a smoke bomb to signal the official beginning of the show. Ignoring air resistance, an object in free fall will fall d feet in t seconds, where d and t are related by the algebraic model $d = 16t^2$.
(a) How long will it take the bomb to fall 180 feet?
(b) If the smoke bomb is in free fall for 12.5 seconds after it is dropped, how high was the airplane when the smoke bomb was dropped?
23. **Physics Equipment** A physics student obtains the following data involving a ball rolling down an inclined plane, where t is the elapsed time in seconds and y is the distance traveled in inches.

t	0	1	2	3	4	5
y	0	1.2	4.8	10.8	19.2	30

Find an algebraic model that fits the data.

24. **U.S. Air Travel** The number of revenue passengers enplaned in the U.S. over the 14-year period from 1991 to 2004 is shown in Table 1.7.



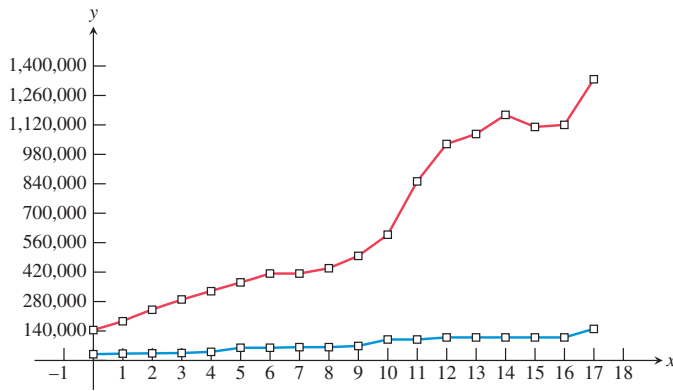
Table 1.7 U.S. Air Travel

Year	Passengers (millions)	Year	Passengers (millions)
1991	452.3	1998	612.9
1992	475.1	1999	636.0
1993	488.5	2000	666.1
1994	528.8	2001	622.1
1995	547.8	2002	612.9
1996	581.2	2003	646.3
1997	594.7	2004	697.8

Source: www.airlines.org

- (a) Graph a scatter plot of the data. Let x be the number of years since 1991.
- (b) From 1991 to 2000 the data seem to follow a linear model. Use the 1991 and 2000 points to find an equation of the line and superimpose the line on the scatter plot.
- (c) According to the linear model, in what year did the number of passengers seem destined to reach 900 million?
- (d) What happened to disrupt the linear model?

Exercises 25–28 refer to the graph below, which shows the *minimum* salaries in major league baseball over a recent 18-year period and the *average* salaries in major league baseball over the same period. Salaries are measured in dollars and time is measured after the starting year (year 0).



Source: Major League Baseball Players Association.

25. Which line is which, and how do you know?
26. After Peter Ueberroth's resignation as baseball commissioner in 1988 and his successor's untimely death in 1989, the team owners broke free of previous restrictions and began an era of competitive spending on player salaries. Identify where the 1990 salaries appear in the graph and explain how you can spot them.
27. The owners attempted to halt the uncontrolled spending by proposing a salary cap, which prompted a players' strike in 1994. The strike caused the 1995 season to be shortened and left many fans angry. Identify where the 1995 salaries appear in the graph and explain how you can spot them.
28. **Writing to Learn** Analyze the general patterns in the graphical model and give your thoughts about what the long-term implications might be for
- the players;
 - the team owners;
 - the baseball fans.

In Exercises 29–38, solve the equation algebraically and graphically.

29. $v^2 - 5 = 8 - 2v^2$
30. $(x + 11)^2 = 121$
31. $2x^2 - 5x + 2 = (x - 3)(x - 2) + 3x$
32. $x^2 - 7x - \frac{3}{4} = 0$

33. $x(2x - 5) = 12$
34. $x(2x - 1) = 10$
35. $x(x + 7) = 14$
36. $x^2 - 3x + 4 = 2x^2 - 7x - 8$
37. $x + 1 - 2\sqrt{x + 4} = 0$
38. $\sqrt{x} + x = 1$

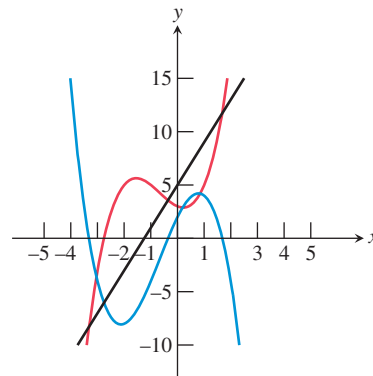
In Exercises 39–46, solve the equation graphically by converting it to an equivalent equation with 0 on the right-hand side and then finding the x -intercepts.

39. $2x - 5 = \sqrt{x + 4}$
40. $|3x - 2| = 2\sqrt{x + 8}$
41. $|2x - 5| = 4 - |x - 3|$
42. $\sqrt{x + 6} = 6 - 2\sqrt{5 - x}$
43. $2x - 3 = x^3 - 5$
44. $x + 1 = x^3 - 2x - 5$
45. $(x + 1)^{-1} = x^{-1} + x$
46. $x^2 = |x|$
47. Swan Auto Rental charges \$32 per day plus \$0.18 per mile for an automobile rental.

- Elaine rented a car for one day and she drove 83 miles. How much did she pay?
- Ramon paid \$69.80 to rent a car for one day. How far did he drive?

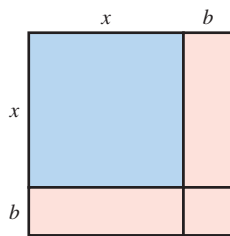
48. **Connecting Graphs and Equations** The curves on the graph below are the graphs of the three curves given by

$$\begin{aligned} y_1 &= 4x + 5 \\ y_2 &= x^3 + 2x^2 - x + 3 \\ y_3 &= -x^3 - 2x^2 + 5x + 2. \end{aligned}$$

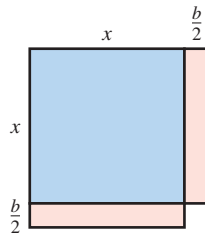


- Write an equation that can be solved to find the points of intersection of the graphs of y_1 and y_2 .
- Write an equation that can be solved to find the x -intercepts of the graph of y_3 .
- Writing to Learn** How does the graphical model reflect the fact that the answers to (a) and (b) are equivalent algebraically?
- Confirm numerically that the x -intercepts of y_3 give the same values when substituted into the expressions for y_1 and y_2 .

- 49. Exploring Grapher Failure** Let $y = (x^{200})^{1/200}$.
- (a) Explain algebraically why $y = x$ for all $x \geq 0$.
 - (b) Graph the equation $y = (x^{200})^{1/200}$ in the window $[0, 1]$ by $[0, 1]$.
 - (c) Is the graph different from the graph of $y = x$?
 - (d) Can you explain why the grapher failed?
- 50. Connecting Algebra and Geometry** Explain how the algebraic equation $(x + b)^2 = x^2 + 2bx + b^2$ models the areas of the regions in the geometric figure shown below on the left:

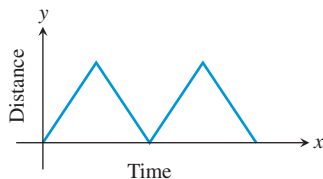


(Ex. 50)



(Ex. 52)

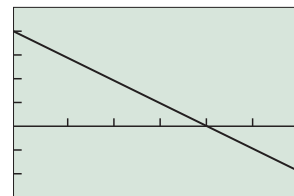
- 51. Exploring Hidden Behavior** Solving graphically, find all real solutions to the following equations. Watch out for hidden behavior.
- (a) $y = 10x^3 + 7.5x^2 - 54.85x + 37.95$
 - (b) $y = x^3 + x^2 - 4.99x + 3.03$
- 52. Connecting Algebra and Geometry** The geometric figure shown on the right above is a large square with a small square missing.
- (a) Find the area of the figure.
 - (b) What area must be added to complete the large square?
 - (c) Explain how the algebraic formula for completing the square models the completing of the square in (b).
- 53. Proving a Theorem** Prove that if n is a positive integer, then $n^2 + 2n$ is either odd or a multiple of 4. Compare your proof with those of your classmates.
- 54. Writing to Learn** The graph below shows the distance from home against time for a jogger. Using information from the graph, write a paragraph describing the jogger's workout.



Standardized Test Questions

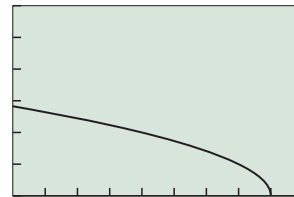
- 55. True or False** A product of real numbers is zero if and only if every factor in the product is zero. Justify your answer.
- 56. True or False** An algebraic model can always be used to make accurate predictions.
- In Exercises 57–60, you may use a graphing calculator to decide which algebraic model corresponds to the given graphical or numerical model.
- (A) $y = 2x + 3$
 - (B) $y = x^2 + 5$
 - (C) $y = 12 - 3x$
 - (D) $y = 4x + 3$
 - (E) $y = \sqrt{8 - x}$

- 57. Multiple Choice**



$[0, 6]$ by $[-9, 15]$

- 58. Multiple Choice**



$[0, 9]$ by $[0, 6]$

- 59. Multiple Choice**

x	1	2	3	4	5	6
y	6	9	14	21	30	41

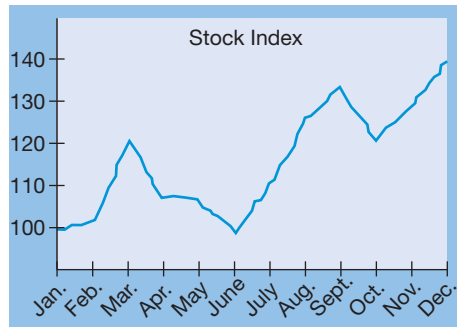
- 60. Multiple Choice**

x	0	2	4	6	8	10
y	3	7	11	15	19	23

Explorations

- 61. Analyzing the Market** Both Ahmad and LaToya watch the stock market throughout the year for stocks that make significant jumps from one month to another. When they spot one, each buys 100 shares. Ahmad's rule is to sell the stock if it fails to perform well for three months in a row. LaToya's rule is to sell in December if the stock has failed to perform well since its purchase.

The graph below shows the monthly performance in dollars (Jan–Dec) of a stock that both Ahmad and LaToya have been watching.



- (a) Both Ahmad and LaToya bought the stock early in the year. In which month?
- (b) At approximately what price did they buy the stock?
- (c) When did Ahmad sell the stock?
- (d) How much did Ahmad lose on the stock?
- (e) **Writing to Learn** Explain why LaToya's strategy was better than Ahmad's for this particular stock in this particular year.
- (f) Sketch a 12-month graph of a stock's performance that would favor Ahmad's strategy over LaToya's.

62. Group Activity Creating Hidden Behavior

You can create your own graphs with hidden behavior. Working in groups of two or three, try this exploration.

- (a) Graph the equation $y = (x + 2)(x^2 - 4x + 4)$ in the window $[-4, 4]$ by $[-10, 10]$.
- (b) Confirm algebraically that this function has zeros only at $x = -2$ and $x = 2$.
- (c) Graph the equation $y = (x + 2)(x^2 - 4x + 4.01)$ in the window $[-4, 4]$ by $[-10, 10]$.
- (d) Confirm algebraically that this function has only one zero, at $x = -2$. (Use the discriminant.)
- (e) **Graph the equation** $(x + 2)(x^2 - 4x + 3.99)$ in the window $[-4, 4]$ by $[-10, 10]$.
- (f) Confirm algebraically that this function has three zeros. (Use the discriminant.)

Extending the Ideas

- 63. The Proliferation of Cell Phones** Table 1.8 shows the number of cellular phone subscribers in the U.S. and their average monthly bill in the years from 1998 to 2004.



Table 1.8 Cellular Phone Subscribers

Year	Subscribers (millions)	Average Local Monthly Bill (\$)
1998	69.2	39.43
1999	86.0	41.24
2000	109.5	45.27
2001	128.4	47.37
2002	140.8	48.40
2003	158.7	49.91
2004	180.4	50.64

Source: Cellular Telecommunication & Internet Association.

- (a) Graph the scatter plots of the number of subscribers and the average local monthly bill as functions of time, letting time t = the number of years after 1990.
 - (b) One of the scatter plots clearly suggests a linear model in the form $y = mx + b$. Use the points at $t = 8$ and $t = 14$ to find a linear model.
 - (c) Superimpose the graph of the linear model onto the scatter plot. Does the fit appear to be good?
 - (d) Why does a linear model seem inappropriate for the other scatter plot? Can you think of a function that might fit the data better?
 - (e) In 1995 there were 33.8 million subscribers with an average local monthly bill of \$51.00. Add these points to the scatter plots.
 - (f) **Writing to Learn** The 1995 points do not seem to fit the models used to represent the 1998–2004 data. Give a possible explanation for this.
- 64. Group Activity** (Continuation of Exercise 63) Discuss the economic forces suggested by the two models in Exercise 63 and speculate about the future by analyzing the graphs.

1.2

Functions and Their Properties

What you'll learn about

- Function Definition and Notation
- Domain and Range
- Continuity
- Increasing and Decreasing Functions
- Boundedness
- Local and Absolute Extrema
- Symmetry
- Asymptotes
- End Behavior

... and why

Functions and graphs form the basis for understanding the mathematics and applications you will see both in your work place and in coursework in college.

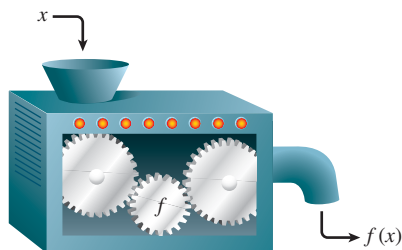


FIGURE 1.9 A “machine” diagram for a function.

A BIT OF HISTORY

The word function in its mathematical sense is generally attributed to Gottfried Leibniz (1646–1716), one of the pioneers in the methods of calculus. His attention to clarity of notation is one of his greatest contributions to scientific progress, which is why we still use his notation in calculus courses today. Ironically, it was not Leibniz but Leonhard Euler (1707–1783) who introduced the familiar notation $f(x)$.

In this section we will introduce the terminology that is used to describe functions throughout this book. Feel free to skim over parts with which you are already familiar, but take the time to become comfortable with concepts that might be new to you (like continuity and symmetry). Even if it takes several days to cover this section, it will be precalculus time well spent.

Function Definition and Notation

Mathematics and its applications abound with examples of formulas by which quantitative variables are related to each other. The language and notation of functions is ideal for that purpose. A function is actually a simple concept; if it were not, history would have replaced it with a simpler one by now. Here is the definition.

DEFINITION Function, Domain, and Range

A function from a set D to a set R is a rule that assigns to every element in D a unique element in R . The set D of all input values is the **domain** of the function, and the set R of all output values is the **range** of the function.

There are many ways to look at functions. One of the most intuitively helpful is the “machine” concept (Figure 1.9), in which values of the domain (x) are fed into the machine (the function f) to produce range values (y). To indicate that y comes from the function acting on x , we use Euler’s elegant **function notation** $y = f(x)$ (which we read as “ **y equals f of x** ” or “**the value of f at x** ”). Here x is the **independent variable** and y is the **dependent variable**.

A function can also be viewed as a **mapping** of the elements of the domain onto the elements of the range. Figure 1.10a shows a function that maps elements from the domain X onto elements of the range Y . Figure 1.10b shows another such mapping, but *this one is not a function*, since the rule does not assign the element x_1 to a *unique* element of Y .

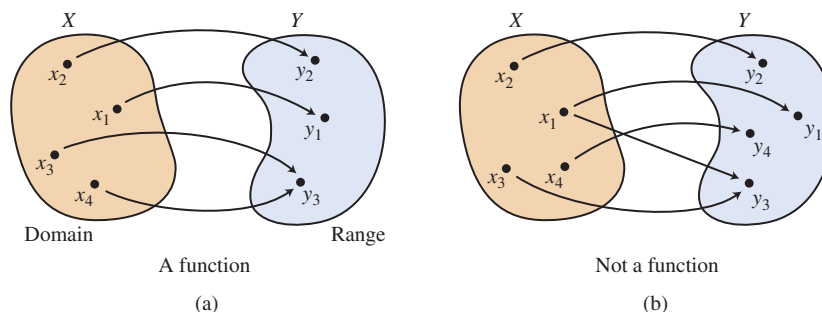


FIGURE 1.10 The diagram in (a) depicts a mapping from X to Y that is a function. The diagram in (b) depicts a mapping from X to Y that is not a function.

This uniqueness of the range value is very important to us as we study function behavior. Knowing that $f(2) = 8$ tells us something about f , and that understanding would be contradicted if we were to discover later that $f(2) = 4$. That is why you will never see a function defined by an ambiguous formula like $f(x) = 3x \pm 2$.

EXAMPLE 1 Defining a Function

Does the formula $y = x^2$ define y as a function of x ?

SOLUTION Yes, y is a function of x . In fact, we can write the formula in function notation: $f(x) = x^2$. When a number x is substituted into the function, the square of x will be the output, and there is no ambiguity about what the square of x is.

Now try Exercise 3.

Another useful way to look at functions is graphically. The **graph of the function $y = f(x)$** is the set of all points $(x, f(x))$, x in the domain of f . We match domain values along the x -axis with their range values along the y -axis to get the ordered pairs that yield the graph of $y = f(x)$.

EXAMPLE 2 Seeing a Function Graphically

Of the three graphs shown in Figure 1.11, which is *not* the graph of a function? How can you tell?

SOLUTION The graph in (c) is not the graph of a function. There are three points on the graph with x -coordinate 0, so the graph does not assign a *unique* value to 0. (Indeed, we can see that there are plenty of numbers between -2 and 2 to which the graph assigns multiple values.) The other two graphs do not have a comparable problem because no vertical line intersects either of the other graphs in more than one point. Graphs that pass this *vertical line test* are the graphs of functions.

Now try Exercise 5.

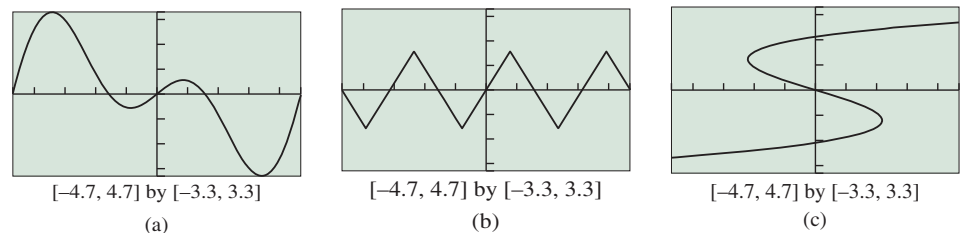


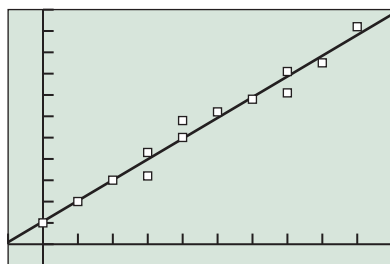
FIGURE 1.11 One of these is not the graph of a function. (Example 2)

Vertical Line Test

A graph (set of points (x, y)) in the xy -plane defines y as a function of x if and only if no vertical line intersects the graph in more than one point.

WHAT ABOUT DATA?

When moving from a numerical model to an algebraic model we will often use a function to approximate data pairs that by themselves violate our definition. In Figure 1.12 we can see that several pairs of data points fail the vertical line test, and yet the linear function approximates the data quite well.



$[-1, 10]$ by $[-1, 11]$

FIGURE 1.12 The data points fail the vertical line test but are nicely approximated by a linear function.

NOTE

The symbol “ \cup ” is read “union.” It means that the elements of the two sets are combined to form one set.

Domain and Range

We will usually define functions algebraically, giving the rule explicitly in terms of the domain variable. The rule, however, does not tell the complete story without some consideration of what the domain actually is.

For example, we can define the volume of a sphere as a function of its radius by the formula

$$V(r) = \frac{4}{3}\pi r^3 \text{ (Note that this is “V of r”—not “V} \cdot r\text{”).}$$

This *formula* is defined for all real numbers, but the volume *function* is not defined for negative r values. So, if our intention were to study the volume function, we would restrict the domain to be all $r \geq 0$.

Agreement

Unless we are dealing with a model (like volume) that necessitates a restricted domain, we will assume that the domain of a function defined by an algebraic expression is the same as the domain of the algebraic expression, the **implied domain**. For models, we will use a domain that fits the situation, the **relevant domain**.

EXAMPLE 3 Finding the Domain of a Function

Find the domain of each of these functions:

(a) $f(x) = \sqrt{x+3}$

(b) $g(x) = \frac{\sqrt{x}}{x-5}$

(c) $A(s) = (\sqrt{3}/4)s^2$, where $A(s)$ is the area of an equilateral triangle with sides of length s .

SOLUTION**Solve Algebraically**

- (a) The expression under a radical may not be negative. We set $x+3 \geq 0$ and solve to find $x \geq -3$. The domain of f is the interval $[-3, \infty)$.
- (b) The expression under a radical may not be negative; therefore $x \geq 0$. Also, the denominator of a fraction may not be zero; therefore $x \neq 5$. The domain of g is the interval $[0, \infty)$ with the number 5 removed, which we can write as the *union* of two intervals: $[0, 5) \cup (5, \infty)$.
- (c) The algebraic expression has domain all real numbers, but the behavior being modeled restricts s from being negative. The domain of A is the interval $[0, \infty)$.

Support Graphically

We can support our answers in (a) and (b) graphically, as the calculator should not plot points where the function is undefined.

- (a) Notice that the graph of $y = \sqrt{x+3}$ (Figure 1.13a) shows points only for $x \geq -3$, as expected.

continued

- (b) The graph of $y = \sqrt{x}/(x - 5)$ (Figure 1.13b) shows points only for $x \geq 0$, as expected, but shows an unexpected line through the x -axis at $x = 5$. This line, a form of grapher failure described in the previous section, should not be there. Ignoring it, we see that 5, as expected, is not in the domain.
- (c) The graph of $y = (\sqrt{3}/4)s^2$ (Figure 1.13c) shows the unrestricted domain of the algebraic expression: all real numbers. The calculator has no way of knowing that s is the length of a side of a triangle.

Now try Exercise 11.

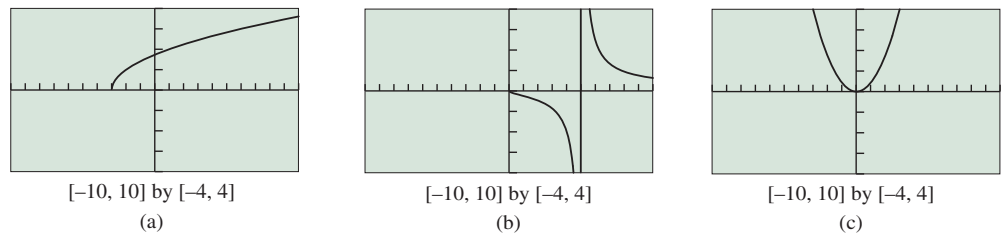


FIGURE 1.13 Graphical support of the algebraic solutions in Example 3. The vertical line in (b) should be ignored because it results from grapher failure. The points in (c) with negative x -coordinates should be ignored because the calculator does not know that x is a length (but we do).

Finding the range of a function algebraically is often much harder than finding the domain, although graphically the things we look for are similar: To find the *domain* we look for all x -coordinates that correspond to points on the graph, and to find the *range* we look for all y -coordinates that correspond to points on the graph. A good approach is to use graphical and algebraic approaches simultaneously, as we show in Example 4.

EXAMPLE 4 Finding the Range of a Function

Find the range of the function $f(x) = \frac{2}{x}$.

SOLUTION

Solve Graphically The graph of $y = \frac{2}{x}$ is shown in Figure 1.14.

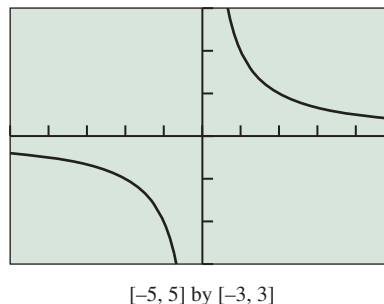


FIGURE 1.14 The graph of $y = 2/x$. Is $y = 0$ in the range?

It appears that $x = 0$ is not in the domain (as expected, because a denominator cannot be zero). It also appears that the range consists of all real numbers except 0.

continued

FUNCTION NOTATION

A grapher typically does not use function notation. So the function $f(x) = x^2 + 1$ is entered as $y_1 = x^2 + 1$. On some graphers you can evaluate f at $x = 3$ by entering $y_1(3)$ on the home screen. On the other hand, on other graphers $y_1(3)$ means $y_1 \cdot 3$.

Confirm Algebraically

We confirm that 0 is not in the range by trying to solve $2/x = 0$:

$$\begin{aligned}\frac{2}{x} &= 0 \\ 2 &= 0 \cdot x \\ 2 &= 0\end{aligned}$$

Since the equation $2 = 0$ is never true, $2/x = 0$ has no solutions, and so $y = 0$ is not in the range. But how do we know that all other real numbers are in the range? We let k be any other real number and try to solve $2/x = k$:

$$\begin{aligned}\frac{2}{x} &= k \\ 2 &= k \cdot x \\ x &= \frac{2}{k}\end{aligned}$$

As you can see, there was no problem finding an x this time, so 0 is the only number not in the range of f . We write the range $(-\infty, 0) \cup (0, \infty)$.

Now try Exercise 17.

You can see that this is considerably more involved than finding a domain, but we are hampered at this point by not having many tools with which to analyze function behavior. We will revisit the problem of finding ranges in Exercise 86, after having developed the tools that will simplify the analysis.

Continuity

One of the most important properties of the majority of functions that model real-world behavior is that they are *continuous*. Graphically speaking, a function is continuous at a point if the graph does not come apart at that point. We can illustrate the concept with a few graphs (Figure 1.15):

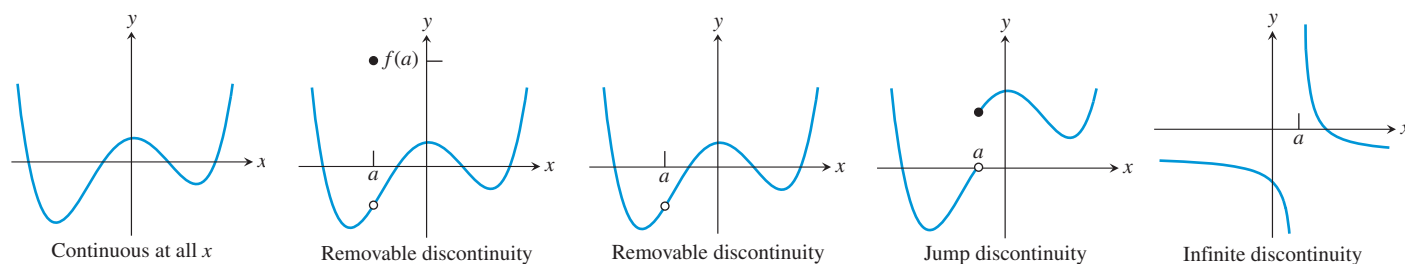
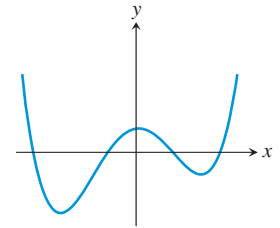


FIGURE 1.15 Some points of discontinuity.

Let's look at these cases individually.

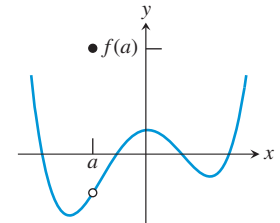


This graph is continuous everywhere. Notice that the graph has no breaks. This means that if we are studying the behavior of the function f for x values close to any particular real number a , we can be assured that the $f(x)$ values will be close to $f(a)$.



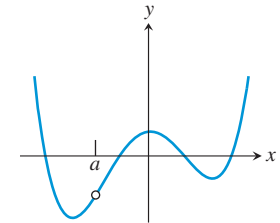
Continuous at all x

This graph is continuous everywhere except for the “hole” at $x = a$. If we are studying the behavior of this function f for x values close to a , we can *not* be assured that the $f(x)$ values will be close to $f(a)$. In this case, $f(x)$ is smaller than $f(a)$ for x near a . This is called a **removable discontinuity** because it can be patched by redefining $f(a)$ so as to plug the hole.



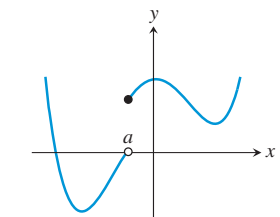
Removable discontinuity

This graph also has a **removable discontinuity** at $x = a$. If we are studying the behavior of this function f for x values close to a , we are still not assured that the $f(x)$ values will be close to $f(a)$, because in this case $f(a)$ doesn't even exist. It is removable because we could define $f(a)$ in such a way as to plug the hole and make f continuous at a .



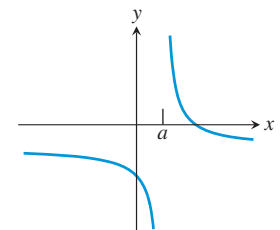
Removable discontinuity

Here is a discontinuity that is not removable. It is a **jump discontinuity** because there is more than just a hole at $x = a$; there is a *jump* in function values that makes the gap impossible to plug with a single point $(a, f(a))$, no matter how we try to redefine $f(a)$.



Jump discontinuity

This is a function with an **infinite discontinuity** at $x = a$. It is definitely not removable.



Infinite discontinuity

The simple geometric concept of an unbroken graph at a point is one of those visual notions that is extremely difficult to communicate accurately in the language of algebra. The key concept from the pictures seems to be that we want the point $(x, f(x))$ to slide smoothly onto the point $(a, f(a))$ without missing it, from either direction. This merely requires that $f(x)$ approach $f(a)$ as a *limit* as x approaches a . A function f is **continuous at $x = a$** if $\lim_{x \rightarrow a} f(x) = f(a)$. A function f is **discontinuous at $x = a$** if it is not continuous at $x = a$.

CHOOSING VIEWING WINDOWS

Most viewing windows will show a vertical line for the function in Figure 1.16. It is sometimes possible to choose a viewing window in which the vertical line does not appear, as we did in Figure 1.16.

EXAMPLE 5 Identifying Points of Discontinuity

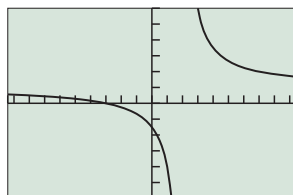
Judging from the graphs, which of the following figures shows functions that are discontinuous at $x = 2$? Are any of the discontinuities removable?

SOLUTION Figure 1.16 shows a function that is undefined at $x = 2$ and hence not continuous there. The discontinuity at $x = 2$ is not removable.

The function graphed in Figure 1.17 is a quadratic polynomial whose graph is a parabola, a graph that has no breaks because its domain includes all real numbers. It is continuous for all x .

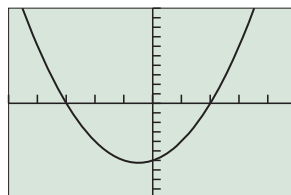
The function graphed in Figure 1.18 is not defined at $x = 2$ and so can not be continuous there. The graph looks like the graph of the line $y = x + 2$, except that there is a hole where the point $(2, 4)$ should be. This is a removable discontinuity.

Now try Exercise 21.



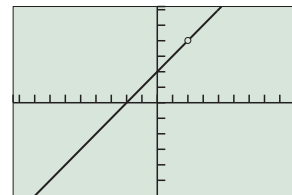
$[-9.4, 9.4]$ by $[-6, 6]$

FIGURE 1.16 $f(x) = \frac{x+3}{x-2}$



$[-5, 5]$ by $[-10, 10]$

FIGURE 1.17 $g(x) = (x+3)(x-2)$



$[-9.4, 9.4]$ by $[-6.2, 6.2]$

FIGURE 1.18 $h(x) = \frac{x^2 - 4}{x - 2}$

Increasing and Decreasing Functions

Another function concept that is easy to understand graphically is the property of being increasing, decreasing, or constant on an interval. We illustrate the concept with a few graphs (Figure 1.19):

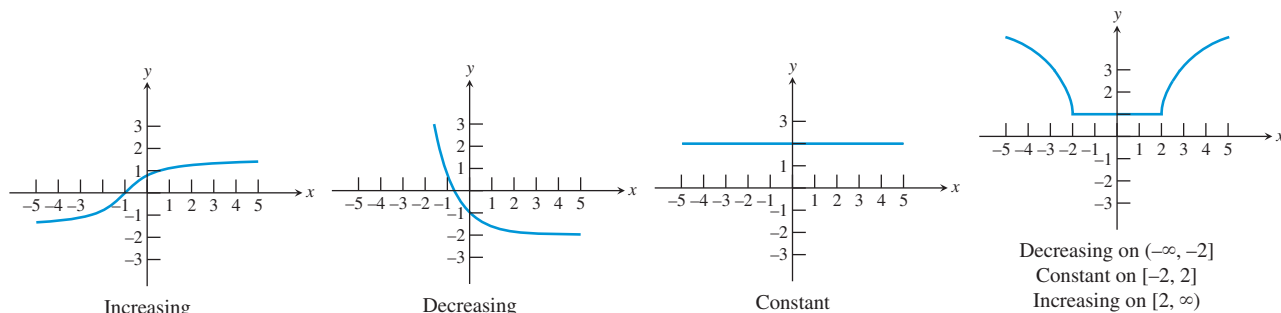


FIGURE 1.19 Examples of increasing, decreasing, or constant on an interval.

Once again the idea is easy to communicate graphically, but how can we identify these properties of functions algebraically? Exploration 1 will help to set the stage for the algebraic definition.

EXPLORATION 1 **Increasing, Decreasing, and Constant Data**

1. Of the three tables of numerical data below, which would be modeled by a function that is (a) increasing, (b) decreasing, (c) constant?

X	Y1	X	Y2	X	Y3
−2	12	−2	3	−2	−5
−1	12	−1	1	−1	−3
0	12	0	0	0	−1
1	12	1	−2	1	1
3	12	3	−6	3	4
7	12	7	−12	7	10

2. Make a list of $\Delta Y1$, the *change* in Y1 values as you move down the list. As you move from $Y1 = a$ to $Y1 = b$, the change is $\Delta Y1 = b - a$. Do the same for the values of Y2 and Y3.

X moves from	ΔX	$\Delta Y1$	X moves from	ΔX	$\Delta Y2$	X moves from	ΔX	$\Delta Y3$
−2 to −1	1		−2 to −1	1		−2 to −1	1	
−1 to 0	1		−1 to 0	1		−1 to 0	1	
0 to 1	1		0 to 1	1		0 to 1	1	
1 to 3	2		1 to 3	2		1 to 3	2	
3 to 7	4		3 to 7	4		3 to 7	4	

3. What is true about the quotients $\Delta Y/\Delta X$ for an increasing function? For a decreasing function? For a constant function?
4. Where else have you seen the quotient $\Delta Y/\Delta X$? Does this reinforce your answers in part 3?

Δ LIST ON A CALCULATOR

Your calculator might be able to help you with the numbers in Exploration 1. Some calculators have a " Δ List" operation that will calculate the changes as you move down a list. For example, the command " Δ List (L1) \rightarrow L3" will store the differences from L1 into L3. Note that Δ List (L1) is always one entry shorter than L1 itself.

Your analysis of the quotients $\Delta Y/\Delta X$ in the exploration should help you to understand the following definition.

DEFINITION Increasing, Decreasing, and Constant Function on an Interval

A function f is **increasing** on an interval if, for any two points in the interval, a positive change in x results in a positive change in $f(x)$.

A function f is **decreasing** on an interval if, for any two points in the interval, a positive change in x results in a negative change in $f(x)$.

A function f is **constant** on an interval if, for any two points in the interval, a positive change in x results in a zero change in $f(x)$.



EXAMPLE 6 Analyzing a Function for Increasing-Decreasing Behavior

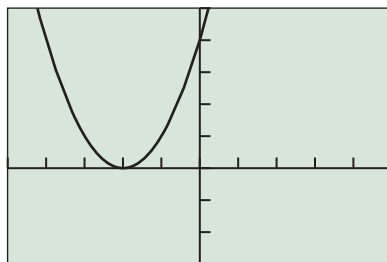
For each function, tell the intervals on which it is increasing and the intervals on which it is decreasing.

(a) $f(x) = (x + 2)^2$ (b) $g(x) = \frac{x^2}{x^2 - 1}$

SOLUTION

Solve Graphically

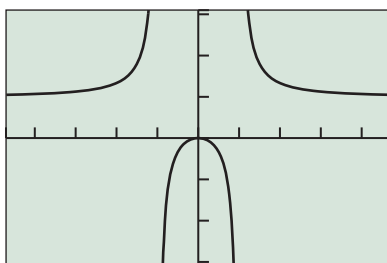
- (a) We see from the graph in Figure 1.20 that f is decreasing on $(-\infty, -2]$ and increasing on $[-2, \infty)$. (Notice that we include -2 in both intervals. Don't worry that this sets up some contradiction about what happens *at* -2 , because we only talk about functions increasing or decreasing on *intervals*, and -2 is not an interval.)



$[-5, 5]$ by $[-3, 5]$

FIGURE 1.20 The function $f(x) = (x + 2)^2$ decreases on $(-\infty, -2]$ and increases on $[-2, \infty)$. (Example 6)

- (b) We see from the graph in Figure 1.21 that g is increasing on $(-\infty, -1)$, increasing again on $(-1, 0]$, decreasing on $[0, 1)$, and decreasing again on $(1, \infty)$.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

FIGURE 1.21 The function $g(x) = x^2/(x^2 - 1)$ increases on $(-\infty, -1)$ and $(-1, 0]$; the function decreases on $[0, 1)$ and $(1, \infty)$. (Example 6)

Now try Exercise 33.

You may have noticed that we are making some assumptions about the graphs. How do we know that they don't turn around somewhere off the screen? We will develop some ways to answer that question later in the book, but the most powerful methods will await you when you study calculus.

Boundedness

The concept of *boundedness* is fairly simple to understand both graphically and algebraically. We will move directly to the algebraic definition after motivating the concept with some typical graphs (Figure 1.22).

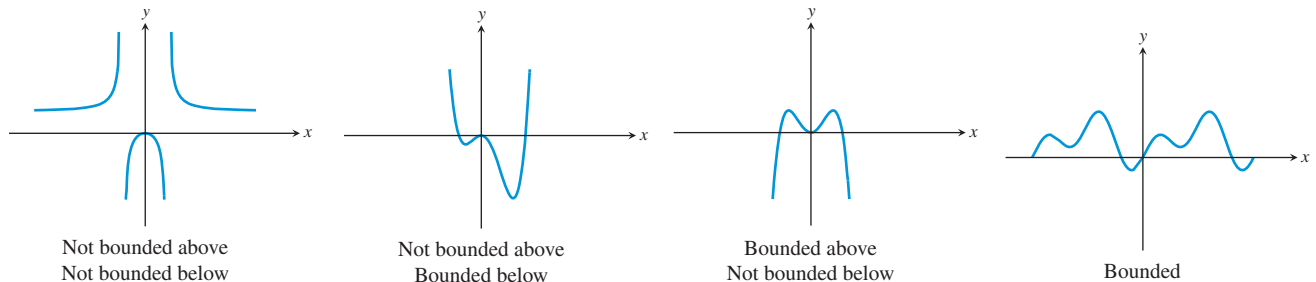


FIGURE 1.22 Some examples of graphs bounded and not bounded above and below.

DEFINITION Lower Bound, Upper Bound, and Bounded

A function f is **bounded below** if there is some number b that is less than or equal to every number in the range of f . Any such number b is called a **lower bound** of f .

A function f is **bounded above** if there is some number B that is greater than or equal to every number in the range of f . Any such number B is called an **upper bound** of f .

A function f is **bounded** if it is bounded both above and below.

We can extend the above definition to the idea of **bounded on an interval** by restricting the domain of consideration in each part of the definition to the interval we wish to consider. For example, the function $f(x) = 1/x$ is bounded above on the interval $(-\infty, 0)$ and bounded below on the interval $(0, \infty)$.

EXAMPLE 7 Checking Boundedness

Identify each of these functions as bounded below, bounded above, or bounded.

(a) $w(x) = 3x^2 - 4$

(b) $p(x) = \frac{x}{1 + x^2}$

SOLUTION

Solve Graphically

The two graphs are shown in Figure 1.23. It appears that w is bounded below, and p is bounded.

Confirm Graphically

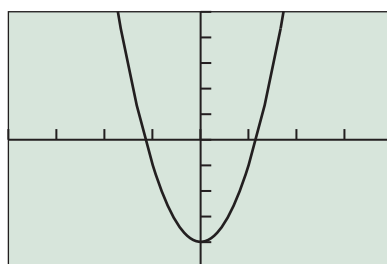
We can confirm that w is bounded below by finding a lower bound, as follows:

$$\begin{aligned} x^2 &\geq 0 && x^2 \text{ is nonnegative} \\ 3x^2 &\geq 0 && \text{Multiply by 3.} \\ 3x^2 - 4 &\geq 0 - 4 && \text{Subtract 4.} \\ 3x^2 - 4 &\geq -4 \end{aligned}$$

Thus, -4 is a lower bound for $w(x) = 3x^2 - 4$.

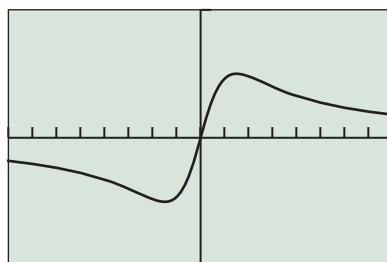
We leave the verification that p is bounded as an exercise (Exercise 77).

Now try Exercise 37.



$[-4, 4]$ by $[-5, 5]$

(a)



$[-8, 8]$ by $[-1, 1]$

(b)

FIGURE 1.23 The graphs for Example 7. Which are bounded where?

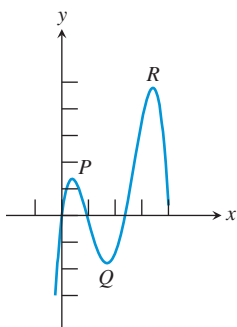


FIGURE 1.24 The graph suggests that f has a local maximum at P , a local minimum at Q , and a local maximum at R .

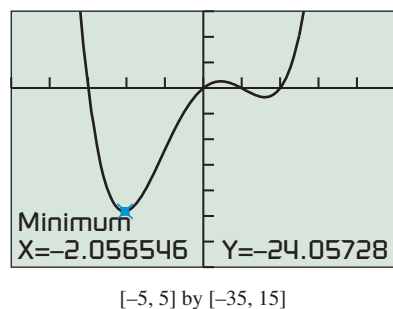


FIGURE 1.25 A graph of $y = x^4 - 7x^2 + 6x$. (Example 8)

USING A GRAPHER TO FIND LOCAL EXTREMA

Most modern graphers have built in “maximum” and “minimum” finders that identify local extrema by looking for sign changes in Δy . It is not easy to find local extrema by zooming in on them, as the graphs tend to flatten out and hide the very behavior you are looking at. If you use this method, keep narrowing the vertical window to maintain some curve in the graph.

Local and Absolute Extrema

Many graphs are characterized by peaks and valleys where they change from increasing to decreasing and vice versa. The extreme values of the function (or *local extrema*) can be characterized as either *local maxima* or *local minima*. The distinction can be easily seen graphically. Figure 1.24 shows a graph with three local extrema: local maxima at points P and R and a local minimum at Q .

This is another function concept that is easier to see graphically than to describe algebraically. Notice that a local maximum does not have to be *the* maximum value of a function; it only needs to be the maximum value of the function on *some* tiny interval.

We have already mentioned that the best method for analyzing increasing and decreasing behavior involves calculus. Not surprisingly, the same is true for local extrema. We will generally be satisfied in this course with approximating local extrema using a graphing calculator, although sometimes an algebraic confirmation will be possible when we learn more about specific functions.

DEFINITION Local and Absolute Extrema

A **local maximum** of a function f is a value $f(c)$ that is greater than or equal to all range values of f on some open interval containing c . If $f(c)$ is greater than or equal to all range values of f , then $f(c)$ is the **maximum** (or **absolute maximum**) value of f .

A **local minimum** of a function f is a value $f(c)$ that is less than or equal to all range values of f on some open interval containing c . If $f(c)$ is less than or equal to all range values of f , then $f(c)$ is the **minimum** (or **absolute minimum**) value of f .

Local extrema are also called **relative extrema**.

EXAMPLE 8 Identifying Local Extrema

Decide whether $f(x) = x^4 - 7x^2 + 6x$ has any local maxima or local minima. If so, find each local maximum value or minimum value and the value of x at which each occurs.

SOLUTION The graph of $y = x^4 - 7x^2 + 6x$ (Figure 1.25) suggests that there are two local minimum values and one local maximum value. We use the graphing calculator to approximate local minima as -24.06 (which occurs at $x \approx -2.06$) and -1.77 (which occurs at $x \approx 1.60$). Similarly, we identify the (approximate) local maximum as 1.32 (which occurs at $x \approx 0.46$).

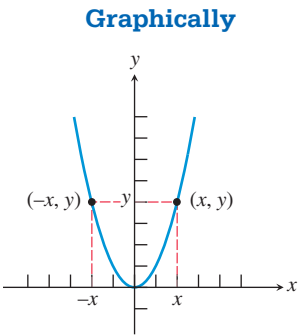
Now try Exercise 41.

Symmetry

In the graphical sense, the word “symmetry” in mathematics carries essentially the same meaning as it does in art: The picture (in this case, the graph) “looks the same” when viewed in more than one way. The interesting thing about mathematical symmetry is that it can be characterized numerically and algebraically as well. We will be

looking at three particular types of symmetry, each of which can be spotted easily from a graph, a table of values, or an algebraic formula, once you know what to look for. Since it is the connections among the three models (graphical, numerical, and algebraic) that we need to emphasize in this section, we will illustrate the various symmetries in all three ways, side-by-side.

Symmetry with respect to the y-axis
Example: $f(x) = x^2$



Numerically

x	$f(x)$
-3	9
-2	4
-1	1
1	1
2	4
3	9

Algebraically

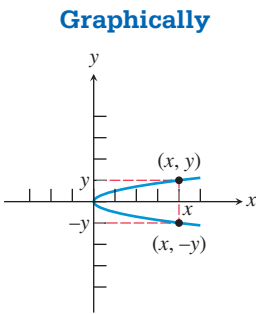
For all x in the domain of f ,

$$f(-x) = f(x)$$

Functions with this property (for example, x^n , n even) are **even** functions.

FIGURE 1.26 The graph looks the same to the left of the y -axis as it does to the right of it.

Symmetry with respect to the x-axis
Example: $x = y^2$



Numerically

x	y
9	-3
4	-2
1	-1
1	1
4	2
9	3

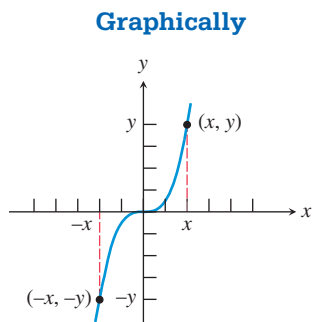
Algebraically

Graphs with this kind of symmetry are not functions (except the zero function), but we can say that $(x, -y)$ is on the graph whenever (x, y) is on the graph.

FIGURE 1.27 The graph looks the same above the x -axis as it does below it.

Symmetry with respect to the origin

Example: $f(x) = x^3$



Numerically

x	y
-3	-27
-2	-8
-1	-1
1	1
2	8
3	27

Algebraically

For all x in the domain of f ,

$$f(-x) = -f(x).$$

Functions with this property (for example, x^n , n odd) are **odd** functions.

FIGURE 1.28 The graph looks the same upside-down as it does rightside-up.

EXAMPLE 9 Checking Functions for Symmetry

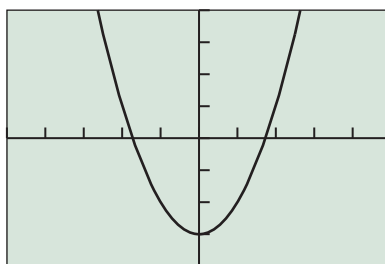
Tell whether each of the following functions is odd, even, or neither.

(a) $f(x) = x^2 - 3$ (b) $g(x) = x^2 - 2x - 2$ (c) $h(x) = \frac{x^3}{4 - x^2}$

SOLUTION

(a) Solve Graphically

The graphical solution is shown in Figure 1.29.



$[-5, 5]$ by $[-4, 4]$

FIGURE 1.29 This graph appears to be symmetric with respect to the y -axis, so we conjecture that f is an even function.

Confirm Algebraically

We need to verify that

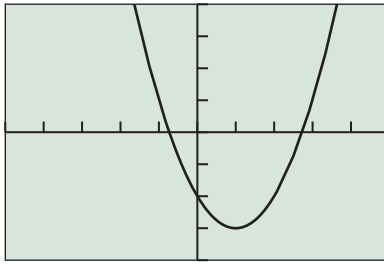
$$f(-x) = f(x)$$

for all x in the domain of f .

$$\begin{aligned} f(-x) &= (-x)^2 - 3 = x^2 - 3 \\ &= f(x) \end{aligned}$$

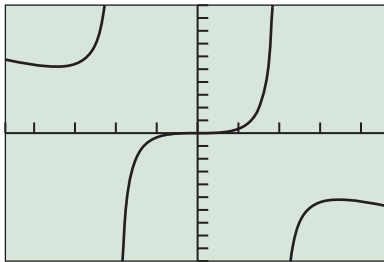
Since this identity is true for all x , the function f is indeed even.

continued



$[-5, 5]$ by $[-4, 4]$

FIGURE 1.30 This graph does not appear to be symmetric with respect to either the y -axis or the origin, so we conjecture that g is neither even nor odd.



$[-4.7, 4.7]$ by $[-10, 10]$

FIGURE 1.31 This graph appears to be symmetric with respect to the origin, so we conjecture that h is an odd function.

(b) Solve Graphically

The graphical solution is shown in Figure 1.30.

Confirm Algebraically

We need to verify that

$$g(-x) \neq g(x) \text{ and } g(-x) \neq -g(x).$$

$$\begin{aligned} g(-x) &= (-x)^2 - 2(-x) - 2 \\ &= x^2 + 2x - 2 \end{aligned}$$

$$\begin{aligned} g(x) &= x^2 - 2x - 2 \\ -g(x) &= -x^2 + 2x + 2 \end{aligned}$$

So $g(-x) \neq g(x)$ and $g(-x) \neq -g(x)$.

We conclude that g is neither odd nor even.

(c) Solve Graphically

The graphical solution is shown in Figure 1.31.

Confirm Algebraically

We need to verify that

$$h(-x) = -h(x)$$

for all x in the domain of h .

$$\begin{aligned} h(-x) &= \frac{(-x)^3}{4 - (-x)^2} = \frac{-x^3}{4 - x^2} \\ &= -h(x) \end{aligned}$$

Since this identity is true for all x except ± 2 (which are not in the domain of h), the function h is odd.

Now try Exercise 49.

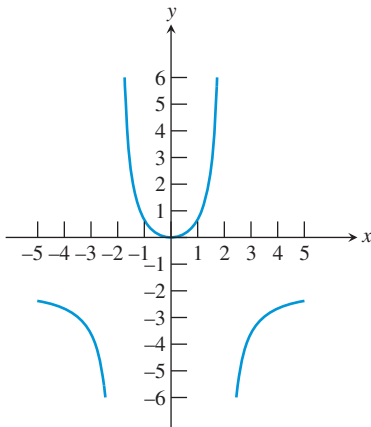


FIGURE 1.32 The graph of $f(x) = 2x^2/(4 - x^2)$ has two vertical asymptotes and one horizontal asymptote.

Asymptotes

Consider the graph of the function $f(x) = \frac{2x^2}{4 - x^2}$ in Figure 1.32.

The graph appears to flatten out to the right and to the left, getting closer and closer to the horizontal line $y = -2$. We call this line a *horizontal asymptote*. Similarly, the graph appears to flatten out as it goes off the top and bottom of the screen, getting closer and closer to the vertical lines $x = -2$ and $x = 2$. We call these lines *vertical asymptotes*. If we superimpose the asymptotes onto Figure 1.32 as dashed lines, you can see that they form a kind of template that describes the limiting behavior of the graph (Figure 1.33 on the next page).

Since asymptotes describe the behavior of the graph at its horizontal or vertical extremities, the definition of an asymptote can best be stated with limit notation. In this definition, note that $x \rightarrow a^-$ means “ x approaches a from the left,” while $x \rightarrow a^+$ means “ x approaches a from the right.”

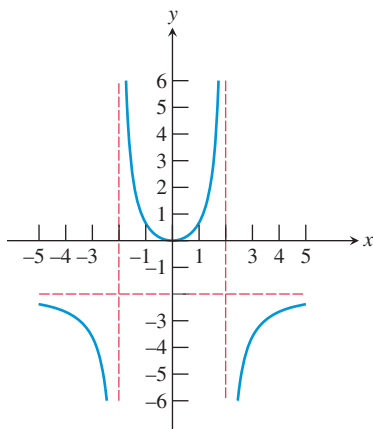
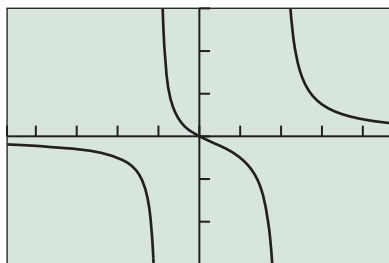


FIGURE 1.33 The graph of $f(x) = 2x^2/(4 - x^2)$ with the asymptotes shown as dashed lines.



$[-4.7, 4.7]$ by $[-3, 3]$

FIGURE 1.34 The graph of $y = x/(x^2 - x - 2)$ has vertical asymptotes of $x = -1$ and $x = 2$ and a horizontal asymptote of $y = 0$. (Example 10)

DEFINITION Horizontal and Vertical Asymptotes

The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if $f(x)$ approaches a limit of b as x approaches $+\infty$ or $-\infty$.

In limit notation:

$$\lim_{x \rightarrow -\infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow +\infty} f(x) = b.$$

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if $f(x)$ approaches a limit of $+\infty$ or $-\infty$ as x approaches a from either direction.

In limit notation:

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^+} f(x) = \pm\infty.$$



EXAMPLE 10 Identifying the Asymptotes of a Graph

Identify any horizontal or vertical asymptotes of the graph of

$$y = \frac{x}{x^2 - x - 2}.$$

SOLUTION The quotient $x/(x^2 - x - 2) = x/((x + 1)(x - 2))$ is undefined at $x = -1$ and $x = 2$, which makes them likely sites for vertical asymptotes. The graph (Figure 1.34) provides support, showing vertical asymptotes of $x = -1$ and $x = 2$.

For large values of x , the numerator (a large number) is dwarfed by the denominator (a *product* of two large numbers), suggesting that $\lim_{x \rightarrow \infty} x/((x + 1)(x - 2)) = 0$. This would indicate a horizontal asymptote of $y = 0$. The graph (Figure 1.34) provides support, showing a horizontal asymptote of $y = 0$ as $x \rightarrow \infty$. Similar logic suggests that $\lim_{x \rightarrow -\infty} x/((x + 1)(x - 2)) = -0 = 0$, indicating the same horizontal asymptote as $x \rightarrow -\infty$. Again, the graph provides support for this.

Now try Exercise 57.

End Behavior

A horizontal asymptote gives one kind of end behavior for a function because it shows how the function behaves as it goes off toward either “end” of the x -axis. Not all graphs approach lines, but it is helpful to consider what *does* happen “out there.” We illustrate with a few examples.

EXAMPLE 11 Matching Functions Using End Behavior

Match the functions with the graphs in Figure 1.35 by considering end behavior. All graphs are shown in the same viewing window.

(a) $y = \frac{3x}{x^2 + 1}$ (b) $y = \frac{3x^2}{x^2 + 1}$ (c) $y = \frac{3x^3}{x^2 + 1}$ (d) $y = \frac{3x^4}{x^2 + 1}$

continued

TIPS ON ZOOMING

Zooming-out is often a good way to investigate end behavior with a graphing calculator. Here are some useful zooming tips:

- Start with a “square” window.
- Set Xscl and Yscl to zero to avoid fuzzy axes.
- Be sure the zoom factors are both the same. (They will be unless you change them.)

SOLUTION When x is very large, the denominator $x^2 + 1$ in each of these functions is almost the same number as x^2 . If we replace $x^2 + 1$ in each denominator by x^2 and then reduce the fractions, we get the simpler functions

(a) $y = \frac{3}{x}$ (close to $y = 0$ for large x) (b) $y = 3$

(c) $y = 3x$ (d) $y = 3x^2$.

So, we look for functions that have end behavior resembling, respectively, the functions

(a) $y = 0$ (b) $y = 3$ (c) $y = 3x$ (d) $y = 3x^2$.

Graph (iv) approaches the line $y = 0$. Graph (iii) approaches the line $y = 3$. Graph (ii) approaches the line $y = 3x$. Graph (i) approaches the parabola $y = 3x^2$. So, (a) matches (iv), (b) matches (iii), (c) matches (ii), and (d) matches (i).

Now try Exercise 65.

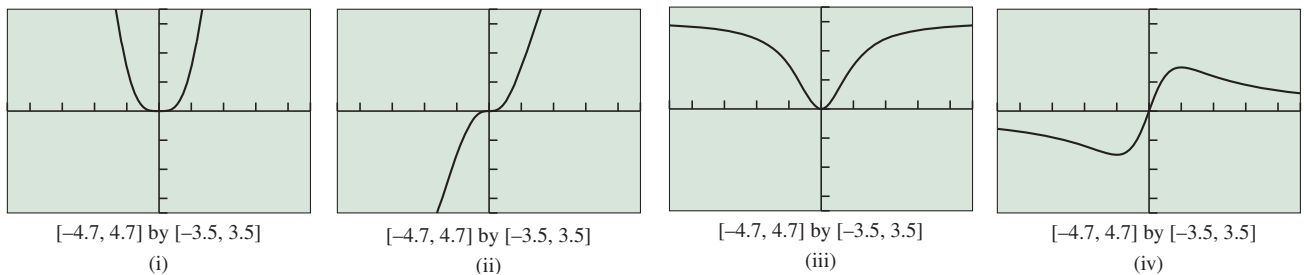


FIGURE 1.35 Match the graphs with the functions in Example 11.

For more complicated functions we are often content with knowing whether the end behavior is bounded or unbounded in either direction.

QUICK REVIEW 1.2 *(For help, go to Sections A.3, P.3, and P.5.)*

In Exercises 1–4, solve the equation or inequality.

1. $x^2 - 16 = 0$
2. $9 - x^2 = 0$
3. $x - 10 < 0$
4. $5 - x \leq 0$

In Exercises 5–10, find all values of x algebraically for which the algebraic expression is *not* defined. Support your answer graphically.

5. $\frac{x}{x - 16}$
6. $\frac{x}{x^2 - 16}$

7. $\sqrt{x - 16}$

8. $\frac{\sqrt{x^2 + 1}}{x^2 - 1}$

9. $\frac{\sqrt{x + 2}}{\sqrt{3 - x}}$

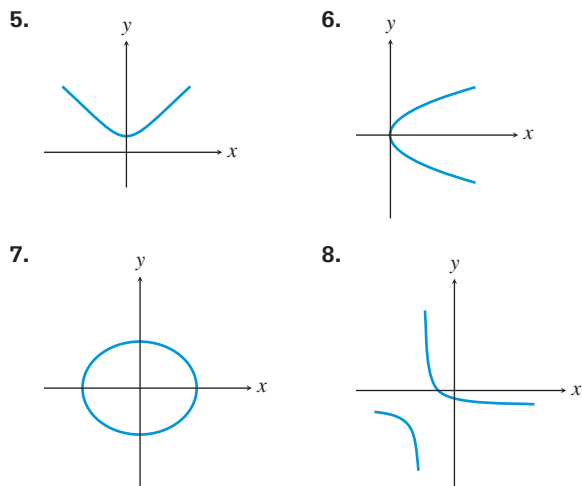
10. $\frac{x^2 - 2x}{x^2 - 4}$

SECTION 1.2 EXERCISES

In Exercises 1–4, determine whether the formula determines y as a function of x . If not, explain why not.

1. $y = \sqrt{x-4}$
2. $y = x^2 \pm 3$
3. $x = 2y^2$
4. $x = 12 - y$

In Exercises 5–8, use the vertical line test to determine whether the curve is the graph of a function.



In Exercises 9–16, find the domain of the function algebraically and support your answer graphically.

9. $f(x) = x^2 + 4$
10. $h(x) = \frac{5}{x-3}$
11. $f(x) = \frac{3x-1}{(x+3)(x-1)}$
12. $f(x) = \frac{1}{x} + \frac{5}{x-3}$
13. $g(x) = \frac{x}{x^2-5x}$
14. $h(x) = \frac{\sqrt{4-x^2}}{x-3}$
15. $h(x) = \frac{\sqrt{4-x}}{(x+1)(x^2+1)}$
16. $f(x) = \sqrt{x^4-16x^2}$

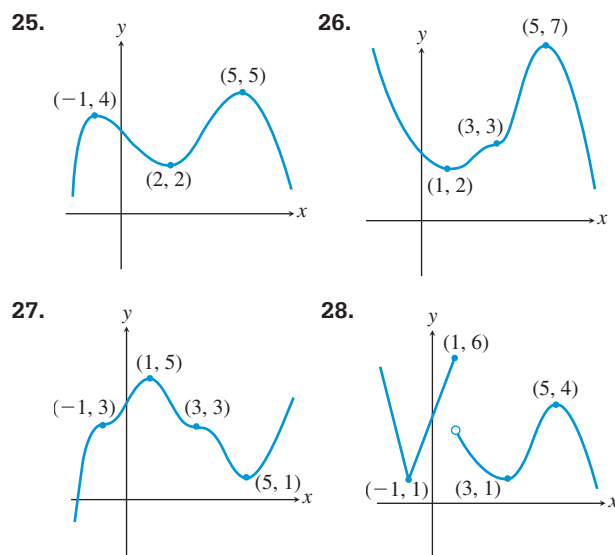
In Exercises 17–20, find the range of the function.

17. $f(x) = 10 - x^2$
18. $g(x) = 5 + \sqrt{4-x}$
19. $f(x) = \frac{x^2}{1-x^2}$
20. $g(x) = \frac{3+x^2}{4-x^2}$

In Exercises 21–24, graph the function and tell whether or not it has a point of discontinuity at $x = 0$. If there is a discontinuity, tell whether it is removable or nonremovable.

21. $g(x) = \frac{3}{x}$
22. $h(x) = \frac{x^3+x}{x}$
23. $f(x) = \frac{|x|}{x}$
24. $g(x) = \frac{x}{x-2}$

In Exercises 25–28, state whether each labeled point identifies a local minimum, a local maximum, or neither. Identify intervals on which the function is decreasing and increasing.



In Exercises 29–34, graph the function and identify intervals on which the function is increasing, decreasing, or constant.

29. $f(x) = |x+2| - 1$
30. $f(x) = |x+1| + |x-1| - 3$
31. $g(x) = |x+2| + |x-1| - 2$
32. $h(x) = 0.5(x+2)^2 - 1$
33. $g(x) = 3 - (x-1)^2$
34. $f(x) = x^3 - x^2 - 2x$

In Exercises 35–40, determine whether the function is bounded above, bounded below, or bounded on its domain.

35. $y = 32$
36. $y = 2 - x^2$
37. $y = 2^x$
38. $y = 2^{-x}$
39. $y = \sqrt{1-x^2}$
40. $y = x - x^3$

In Exercises 41–46, use a grapher to find all local maxima and minima and the values of x where they occur. Give values rounded to two decimal places.

41. $f(x) = 4 - x + x^2$
42. $g(x) = x^3 - 4x + 1$
43. $h(x) = -x^3 + 2x - 3$
44. $f(x) = (x+3)(x-1)^2$
45. $h(x) = x^2\sqrt{x+4}$
46. $g(x) = x|2x+5|$

In Exercises 47–54, state whether the function is odd, even, or neither. Support graphically and confirm algebraically.

47. $f(x) = 2x^4$

48. $g(x) = x^3$

49. $f(x) = \sqrt{x^2 + 2}$

50. $g(x) = \frac{3}{1 + x^2}$

51. $f(x) = -x^2 + 0.03x + 5$

52. $f(x) = x^3 + 0.04x^2 + 3$

53. $g(x) = 2x^3 - 3x$

54. $h(x) = \frac{1}{x}$

In Exercises 55–62, use a method of your choice to find all horizontal and vertical asymptotes of the function.

55. $f(x) = \frac{x}{x-1}$

56. $g(x) = \frac{x-1}{x}$

57. $g(x) = \frac{x+2}{3-x}$

58. $q(x) = 1.5^x$

59. $f(x) = \frac{x^2+2}{x^2-1}$

60. $p(x) = \frac{4}{x^2+1}$

61. $g(x) = \frac{4x-4}{x^3-8}$

62. $h(x) = \frac{2x-4}{x^2-4}$

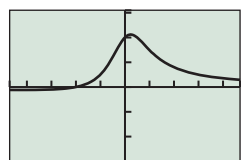
In Exercises 63–66, match the function with the corresponding graph by considering end behavior and asymptotes. All graphs are shown in the same viewing window.

63. $y = \frac{x+2}{2x+1}$

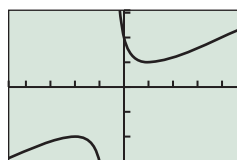
64. $y = \frac{x^2+2}{2x+1}$

65. $y = \frac{x+2}{2x^2+1}$

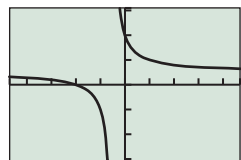
66. $y = \frac{x^3+2}{2x^2+1}$



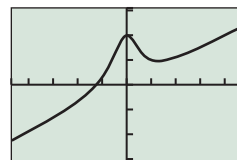
[-4.7, 4.7] by [-3.1, 3.1]
(a)



[-4.7, 4.7] by [-3.1, 3.1]
(c)



[-4.7, 4.7] by [-3.1, 3.1]
(b)



[-4.7, 4.7] by [-3.1, 3.1]
(d)

67. Can a graph cross its own asymptote? The Greek roots of the word “asymptote” mean “not meeting,” since graphs tend to approach, but not meet, their asymptotes. Which of the following functions have graphs that *do* intersect their horizontal asymptotes?

(a) $f(x) = \frac{x}{x^2-1}$

(b) $g(x) = \frac{x}{x^2+1}$

(c) $h(x) = \frac{x^2}{x^3+1}$

68. Can a graph have two horizontal asymptotes?

Although most graphs have at most one horizontal asymptote, it is possible for a graph to have more than one. Which of the following functions have graphs with more than one horizontal asymptote?

(a) $f(x) = \frac{|x^3+1|}{8-x^3}$

(b) $g(x) = \frac{|x-1|}{x^2-4}$

(c) $h(x) = \frac{x}{\sqrt{x^2-4}}$

69. Can a graph intersect its own vertical asymptote?

Graph the function $f(x) = \frac{x-|x|}{x^2} + 1$.

(a) The graph of this function does not intersect its vertical asymptote. Explain why it does not.

(b) Show how you can add a single point to the graph of f and get a graph that *does* intersect its vertical asymptote.

(c) Is the graph in (b) the graph of a function?

70. Writing to Learn Explain why a graph cannot have more than two horizontal asymptotes.

Standardized Test Questions

71. True or False The graph of function f is defined as the set of all points $(x, f(x))$ where x is in the domain of f . Justify your answer.

72. True or False A relation that is symmetric with respect to the x -axis cannot be a function. Justify your answer.

In Exercises 73–76, answer the question without using a calculator.

73. Multiple Choice Which function is continuous?

- (A) Number of children enrolled in a particular school as a function of time
- (B) Outdoor temperature as a function of time
- (C) Cost of U.S. postage as a function of the weight of the letter
- (D) Price of a stock as a function of time
- (E) Number of soft drinks sold at a ballpark as a function of outdoor temperature

74. Multiple Choice Which function is *not* continuous?

- (A) Your altitude as a function of time while flying from Reno to Dallas
- (B) Time of travel from Miami to Pensacola as a function of driving speed
- (C) Number of balls that can fit completely inside a particular box as a function of the radius of the balls
- (D) Area of a circle as a function of radius
- (E) Weight of a particular baby as a function of time after birth

- 75. Decreasing Function** Which function is decreasing?
- (A) Outdoor temperature as a function of time
 - (B) The Dow Jones Industrial Average as a function of time
 - (C) Air pressure in the Earth’s atmosphere as a function of altitude
 - (D) World population since 1900 as a function of time
 - (E) Water pressure in the ocean as a function of depth
- 76. Increasing or Decreasing** Which function cannot be classified as either increasing or decreasing?
- (A) Weight of a lead brick as a function of volume
 - (B) Height of a ball that has been tossed upward as a function of time
 - (C) Time of travel from Buffalo to Syracuse as a function of driving speed
 - (D) Area of a square as a function of side length
 - (E) Height of a swinging pendulum as a function of time

Explorations

- 77. Bounded Functions** As promised in Example 7 of this section, we will give you a chance to prove algebraically that $p(x) = x/(1 + x^2)$ is bounded.
- (a) Graph the function and find the smallest integer k that appears to be an upper bound.
 - (b) Verify that $x/(1 + x^2) < k$ by proving the equivalent inequality $kx^2 - x + k > 0$. (Use the quadratic formula to show that the quadratic has no real zeros.)
 - (c) From the graph, find the greatest integer k that appears to be a lower bound.
 - (d) Verify that $x/(1 + x^2) > k$ by proving the equivalent inequality $kx^2 - x + k < 0$.
- 78. Baylor School Grade Point Averages** Baylor School uses a sliding scale to convert the percentage grades on its transcripts to grade point averages (GPAs). Table 1.9 shows the GPA equivalents for selected grades:



Table 1.9 Converting Grades

Grade (x)	GPA (y)
60	0.00
65	1.00
70	2.05
75	2.57
80	3.00
85	3.36
90	3.69
95	4.00
100	4.28

Source: Baylor School College Counselor.

- (a) Considering GPA (y) as a function of percentage grade (x), is it increasing, decreasing, constant, or none of these?
 - (b) Make a table showing the *change* (Δy) in GPA as you move down the list. (See Exploration 1.)
 - (c) Make a table showing the change in Δy as you move down the list. (This is $\Delta \Delta y$.) Considering the *change* (Δy) in GPA as a function of percentage grade (x), is it increasing, decreasing, constant, or none of these?
 - (d) In general, what can you say about the shape of the graph if y is an increasing function of x and Δy is a decreasing function of x ?
 - (e) Sketch the graph of a function y of x such that y is a decreasing function of x and Δy is an increasing function of x .
- 79. Group Activity** Sketch (freehand) a graph of a function f with domain all real numbers that satisfies all of the following conditions:
- (a) f is continuous for all x ;
 - (b) f is increasing on $(-\infty, 0]$ and on $[3, 5]$;
 - (c) f is decreasing on $[0, 3]$ and on $[5, \infty)$;
 - (d) $f(0) = f(5) = 2$;
 - (e) $f(3) = 0$.
- 80. Group Activity** Sketch (freehand) a graph of a function f with domain all real numbers that satisfies all of the following conditions:
- (a) f is decreasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$;
 - (b) f has a nonremovable point of discontinuity at $x = 0$;
 - (c) f has a horizontal asymptote at $y = 1$;
 - (d) $f(0) = 0$;
 - (e) f has a vertical asymptote at $x = 0$.
- 81. Group Activity** Sketch (freehand) a graph of a function f with domain all real numbers that satisfies all of the following conditions:
- (a) f is continuous for all x ;
 - (b) f is an even function;
 - (c) f is increasing on $[0, 2]$ and decreasing on $[2, \infty)$;
 - (d) $f(2) = 3$.
- 82. Group Activity** Get together with your classmates in groups of two or three. Sketch a graph of a function, but do not show it to the other members of your group. Using the language of functions (as in Exercises 79–81), describe your function as completely as you can. Exchange descriptions with the others in your group and see if you can reproduce each other’s graphs.

Extending the Ideas

- 83.** A function that is bounded above has an infinite number of upper bounds, but there is always a *least upper bound*, i.e., an upper bound that is less than all the others. This least upper bound may or may not be in the range of f . For each of the following functions, find the least upper bound and tell whether or not it is in the range of the function.

(a) $f(x) = 2 - 0.8x^2$

(b) $g(x) = \frac{3x^2}{3 + x^2}$

(c) $h(x) = \frac{1 - x}{x^2}$

(d) $p(x) = 2 \sin(x)$

(e) $q(x) = \frac{4x}{x^2 + 2x + 1}$

- 84. Writing to Learn** A continuous function f has domain all real numbers. If $f(-1) = 5$ and $f(1) = -5$, explain why f must have at least one zero in the interval $[-1, 1]$. (This generalizes to a property of continuous functions known as the Intermediate Value Theorem.)

- 85. Proving a Theorem** Prove that the graph of every odd function with domain all real numbers must pass through the origin.

- 86. Finding the Range** Graph the function $f(x) = \frac{3x^2 - 1}{2x^2 + 1}$ in the window $[-6, 6]$ by $[-2, 2]$.

(a) What is the apparent horizontal asymptote of the graph?

(b) Based on your graph, determine the apparent range of f .

(c) Show algebraically that $-1 \leq \frac{3x^2 - 1}{2x^2 + 1} < 1.5$ for all x , thus confirming your conjecture in part (b).

1.3

Twelve Basic Functions

What you'll learn about

- What Graphs Can Tell Us
- Twelve Basic Functions
- Analyzing Functions Graphically

... and why

As you continue to study mathematics, you will find that the twelve basic functions presented here will come up again and again. By knowing their basic properties, you will recognize them when you see them.



What Graphs Can Tell Us

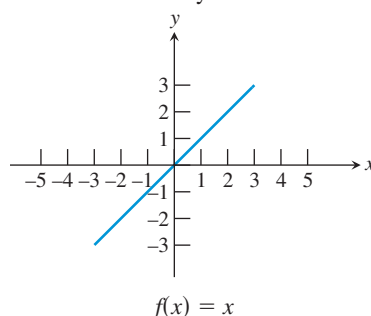
The preceding section has given us a vocabulary for talking about functions and their properties. We have an entire book ahead of us to study these functions in depth, but in this section we want to set the scene by just *looking* at the graphs of twelve “basic” functions that are available on your graphing calculator.

You will find that function attributes such as domain, range, continuity, asymptotes, extrema, increasingness, decreasingness, and end behavior are every bit as graphical as they are algebraic. Moreover, the visual cues are often much easier to spot than the algebraic ones.

In future chapters you will learn more about the algebraic properties that make these functions behave as they do. Only then will you be able to *prove* what is visually apparent in these graphs.

Twelve Basic Functions

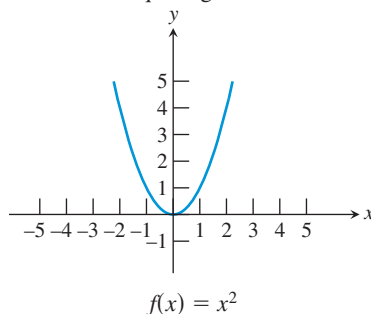
The Identity Function



Interesting fact: This is the only function that acts on every real number by leaving it alone.

FIGURE 1.36

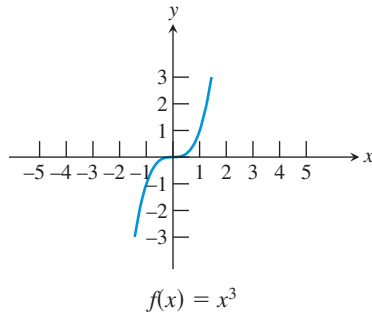
The Squaring Function



Interesting fact: The graph of this function, called a parabola, has a reflection property that is useful in making flashlights and satellite dishes.

FIGURE 1.37

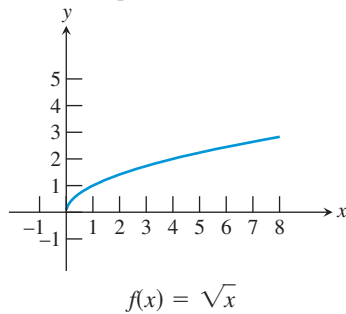
The Cubing Function



Interesting fact: The origin is called a “point of inflection” for this curve because the graph changes curvature at that point.

FIGURE 1.38

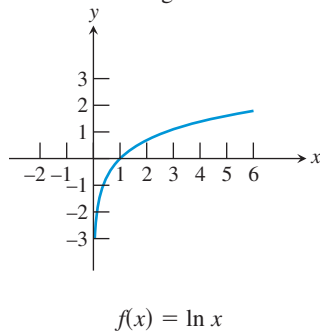
The Square Root Function



Interesting fact: Put any positive number into your calculator. Take the square root. Then take the square root again. Then take the square root again, and so on. Eventually you will always get 1.

FIGURE 1.40

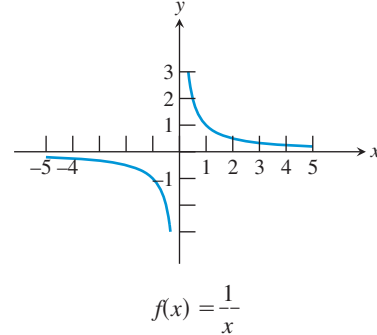
The Natural Logarithm Function



Interesting fact: This function increases very slowly. If the x -axis and y -axis were both scaled with unit lengths of one inch, you would have to travel more than two and a half miles along the curve just to get a foot above the x -axis.

FIGURE 1.42

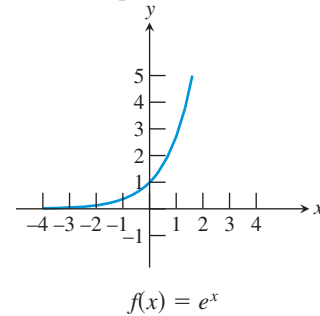
The Reciprocal Function



Interesting fact: This curve, called a hyperbola, also has a reflection property that is useful in satellite dishes.

FIGURE 1.39

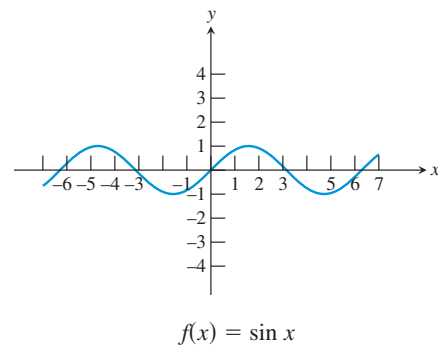
The Exponential Function



Interesting fact: The number e is an irrational number (like π) that shows up in a variety of applications. The symbols e and π were both brought into popular use by the great Swiss mathematician Leonhard Euler (1707–1783).

FIGURE 1.41

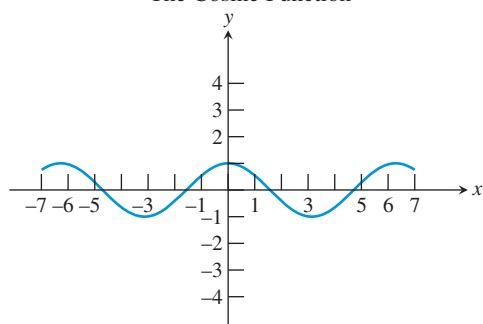
The Sine Function



Interesting fact: This function and the sinus cavities in your head derive their names from a common root: the Latin word for “bay.” This is due to a 12th-century mistake made by Robert of Chester, who translated a word incorrectly from an Arabic manuscript.

FIGURE 1.43

The Cosine Function

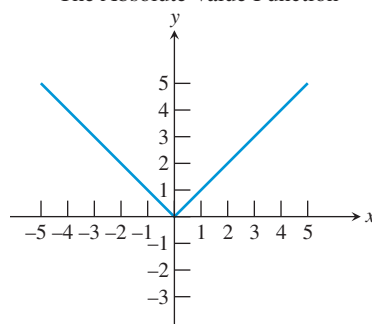


$$f(x) = \cos x$$

Interesting fact: The local extrema of the cosine function occur exactly at the zeros of the sine function, and vice versa.

FIGURE 1.44

The Absolute Value Function

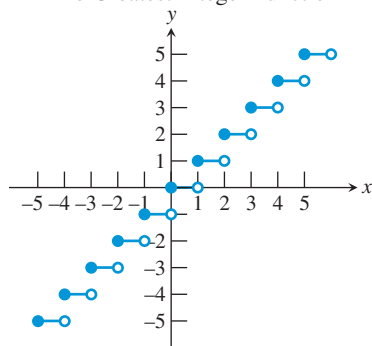


$$f(x) = |x| = \text{abs}(x)$$

Interesting fact: This function has an abrupt change of direction (a “corner”) at the origin, while our other functions are all “smooth” on their domains.

FIGURE 1.45

The Greatest Integer Function

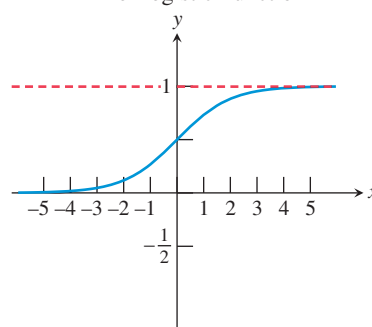


$$f(x) = \text{int}(x)$$

Interesting fact: This function has a jump discontinuity at every integer value of x . Similar-looking functions are called *step functions*.

FIGURE 1.46

The Logistic Function



$$f(x) = \frac{1}{1 + e^{-x}}$$

Interesting fact: There are two horizontal asymptotes, the x -axis and the line $y = 1$. This function provides a model for many applications in biology and business.

FIGURE 1.47

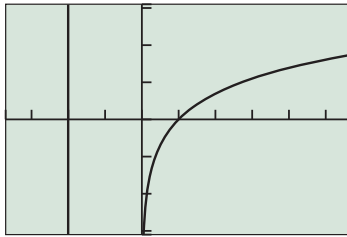
EXAMPLE 1 Looking for Domains

- Nine of the functions have domain the set of all real numbers. Which three do not?
- One of the functions has domain the set of all reals except 0. Which function is it, and why isn't zero in its domain?
- Which two functions have no negative numbers in their domains? Of these two, which one is defined at zero?

SOLUTION

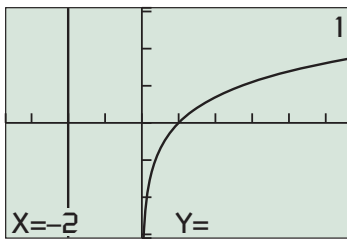
- Imagine dragging a vertical line along the x -axis. If the function has domain the set of all real numbers, then the line will always intersect the graph. The intersection might occur off screen, but the TRACE function on the calculator will show the y -coordinate if there is one. Looking at the graphs in Figures 1.39, 1.40, and 1.42, we conjecture that there are vertical lines that do not intersect the curve.

continued



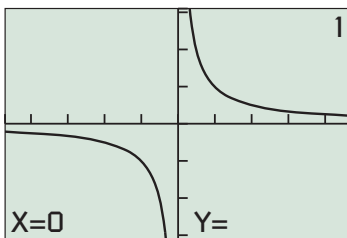
[-3.7, 5.7] by [-3.1, 3.1]

(a)



[-3.7, 5.7] by [-3.1, 3.1]

(b)



[-4.7, 4.7] by [-3.1, 3.1]

(c)

FIGURE 1.48 (a) A vertical line through -2 on the x -axis appears to miss the graph of $y = \ln x$. (b) A TRACE confirms that -2 is not in the domain. (c) A TRACE at $x = 0$ confirms that 0 is not in the domain of $y = 1/x$. (Example 1)

A TRACE at the suspected x -coordinates confirms our conjecture (Figure 1.48). The functions are $y = 1/x$, $y = \sqrt{x}$ and $y = \ln x$.

- (b) The function $y = 1/x$, with a vertical asymptote at $x = 0$, is defined for all real numbers except 0 . This is explained algebraically by the fact that division by zero is undefined.
- (c) The functions $y = \sqrt{x}$ and $y = \ln x$ have no negative numbers in their domains. (We already knew that about the square root function.) While 0 is in the domain of $y = \sqrt{x}$, we can see by tracing that it is not in the domain of $y = \ln x$. We will see the algebraic reason for this in Chapter 3.

Now try Exercise 13.

EXAMPLE 2 Looking for Continuity

Only two of twelve functions have points of discontinuity. Are these points in the domain of the function?

SOLUTION All of the functions have continuous, unbroken graphs except for $y = 1/x$, and $y = \text{int}(x)$.

The graph of $y = 1/x$ clearly has an infinite discontinuity at $x = 0$ (Figure 1.39). We saw in Example 1 that 0 is not in the domain of the function. Since $y = 1/x$ is continuous for every point *in its domain*, it is called a **continuous function**.

The graph of $y = \text{int}(x)$ has a discontinuity at every integer value of x (Figure 1.46). Since this function has discontinuities at points in its domain, it is *not* a continuous function.

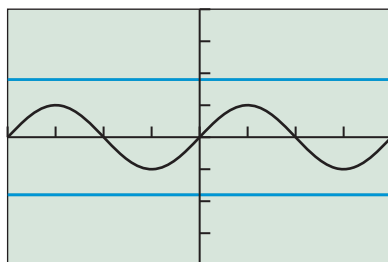
Now try Exercise 15.

EXAMPLE 3 Looking for Boundedness

Only three of the twelve basic functions are bounded (above and below). Which three?

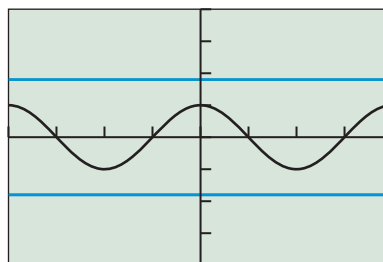
SOLUTION A function that is bounded must have a graph that lies entirely between two horizontal lines. The sine, cosine, and logistic functions have this property (Figure 1.49). It looks like the graph of $y = \sqrt{x}$ might also have this property, but we know that the end behavior of the square root function is unbounded: $\lim_{x \rightarrow \infty} \sqrt{x} = \infty$, so it is really only bounded below. You will learn in Chapter 4 why the sine and cosine functions are bounded.

Now try Exercise 17.



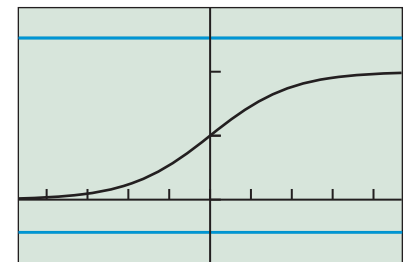
[-2π, 2π] by [-4, 4]

(a)



[-2π, 2π] by [-4, 4]

(b)



[-4.7, 4.7] by [-0.5, 1.5]

(c)

FIGURE 1.49 The graphs of $y = \sin x$, $y = \cos x$, and $y = 1/(1 + e^{-x})$ lie entirely between two horizontal lines and are therefore graphs of bounded functions. (Example 3)

EXAMPLE 4 Looking for Symmetry

Three of the twelve basic functions are even. Which are they?

SOLUTION Recall that the graph of an even function is symmetric with respect to the y -axis. Three of the functions exhibit the required symmetry: $y = x^2$, $y = \cos x$, and $y = |x|$ (Figure 1.50). *Now try Exercise 19.*

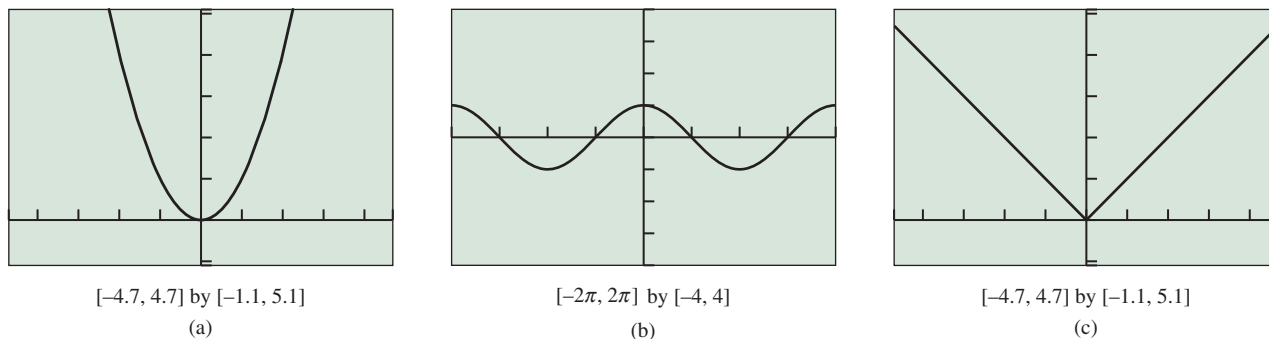


FIGURE 1.50 The graphs of $y = x^2$, $y = \cos x$, and $y = |x|$ are symmetric with respect to the y -axis, indicating that the functions are even. (Example 4)

**Analyzing Functions Graphically**

We could continue to explore the twelve basic functions as in the first four examples, but we also want to make the point that there is no need to restrict ourselves to the basic twelve. We can alter the basic functions slightly and see what happens to their graphs, thereby gaining further visual insights into how functions behave.

**EXAMPLE 5 Analyzing a Function Graphically**

Graph the function $y = (x - 2)^2$. Then answer the following questions:

- (a) On what interval is the function increasing? On what interval is it decreasing?
- (b) Is the function odd, even, or neither?
- (c) Does the function have any extrema?
- (d) How does the graph relate to the graph of the basic function $y = x^2$?

SOLUTION The graph is shown in Figure 1.51.

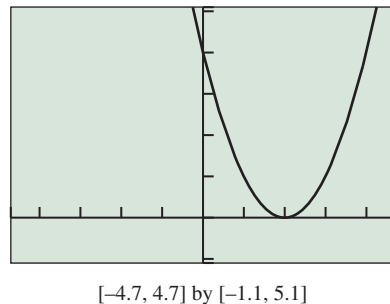


FIGURE 1.51 The graph of $y = (x - 2)^2$. (Example 5)

continued

- (a) The function is increasing if its graph is headed upward as it moves from left to right. We see that it is increasing on the interval $[2, \infty)$. The function is decreasing if its graph is headed downward as it moves from left to right. We see that it is decreasing on the interval $(-\infty, 2]$.
- (b) The graph is not symmetric with respect to the y -axis, nor is it symmetric with respect to the origin. The function is neither.
- (c) Yes, we see that the function has a minimum value of 0 at $x = 2$. (This is easily confirmed by the algebraic fact that $(x - 2)^2 \geq 0$ for all x .)
- (d) We see that the graph of $y = (x - 2)^2$ is just the graph of $y = x^2$ moved two units to the right.

Now try Exercise 35.

EXPLORATION 1 Looking for Asymptotes

- Two of the basic functions have vertical asymptotes at $x = 0$. Which two?
- Form a new function by adding these functions together. Does the new function have a vertical asymptote at $x = 0$?
- Three of the basic functions have horizontal asymptotes at $y = 0$. Which three?
- Form a new function by adding these functions together. Does the new function have a horizontal asymptote at $y = 0$?
- Graph $f(x) = 1/x$, $g(x) = 1/(2x^2 - x)$, and $h(x) = f(x) + g(x)$. Does $h(x)$ have a vertical asymptote at $x = 0$?

EXAMPLE 6 Identifying a Piecewise-Defined Function

Which of the twelve basic functions has the following **piecewise** definition over separate intervals of its domain?

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

SOLUTION You may recognize this as the definition of the absolute value function (Chapter P). Or, you can reason that the graph of this function must look just like the line $y = x$ to the right of the y -axis, but just like the graph of the line $y = -x$ to the left of the y -axis. That is a perfect description of the absolute value graph in Figure 1.45. Either way, we recognize this as a piecewise definition of $f(x) = |x|$.

Now try Exercise 45.

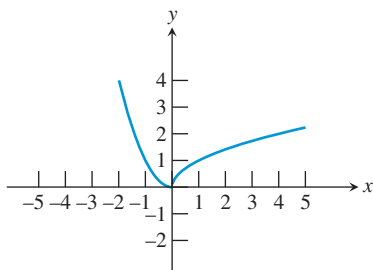
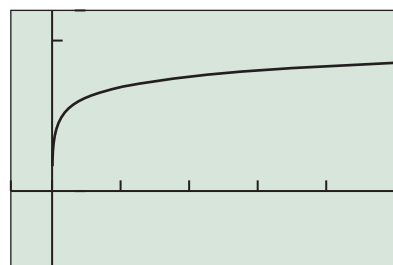


FIGURE 1.52 A piecewise-defined function. (Example 7)



$[-600, 5000]$ by $[-5, 12]$

FIGURE 1.53 The graph of $y = \ln x$ still appears to have a horizontal asymptote, despite the much larger window than in Figure 1.42. (Example 8)

EXAMPLE 7 Defining a Function Piecewise

Using basic functions from this section, construct a piecewise definition for the function whose graph is shown in Figure 1.52. Is your function continuous?

SOLUTION This appears to be the graph of $y = x^2$ to the left of $x = 0$ and the graph of $y = \sqrt{x}$ to the right of $x = 0$. We can therefore define it piecewise as

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$

The function is continuous.

Now try Exercise 47.

You can go a long way toward understanding a function's behavior by looking at its graph. We will continue that theme in the exercises and then revisit it throughout the book. However, you can't go *all* the way toward understanding a function by looking at its graph, as Example 8 shows.

EXAMPLE 8 Looking for a Horizontal Asymptote

Does the graph of $y = \ln x$ (Figure 1.42) have a horizontal asymptote?

SOLUTION In Figure 1.42 it certainly *looks* like there is a horizontal asymptote that the graph is approaching from below. If we choose a much larger window (Figure 1.53), it still looks that way. In fact, we could zoom out on this function all day long and it would *always* look like it is approaching some horizontal asymptote—but it is not. We will show algebraically in Chapter 3 that the end behavior of this function is $\lim_{x \rightarrow \infty} \ln x = \infty$, so its graph must eventually rise above the level of any horizontal line. That rules out any horizontal asymptote, even though there is no *visual* evidence of that fact that we can see by looking at its graph.

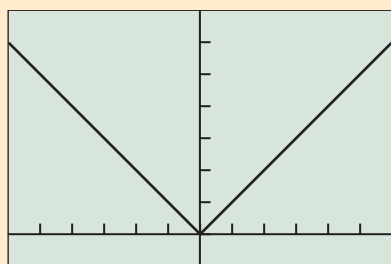
Now try Exercise 55.

EXAMPLE 9 Analyzing a Function

Give a complete analysis of the basic function $f(x) = |x|$.

SOLUTION

BASIC FUNCTION The Absolute Value Function



$[-6, 6]$ by $[-1, 7]$

FIGURE 1.54 The graph of $f(x) = |x|$.

$$f(x) = |x|$$

Domain: All reals

Range: $[0, \infty)$

Continuous

Decreasing on $(-\infty, 0]$; increasing on $[0, \infty)$

Symmetric with respect to the y -axis (an even function)

Bounded below

Local minimum at $(0, 0)$

No horizontal asymptotes

No vertical asymptotes

End behavior: $\lim_{x \rightarrow -\infty} |x| = \infty$ and $\lim_{x \rightarrow \infty} |x| = \infty$

Now try Exercise 67.

QUICK REVIEW 1.3 (For help, go to Sections P.1, P.2, 3.1, and 3.3.)

In Exercises 1–10, evaluate the expression without using a calculator.

1. $|-59.34|$

2. $|5 - \pi|$

3. $|\pi - 7|$

4. $\sqrt{(-3)^2}$

5. $\ln(1)$

7. $(\sqrt[3]{3})^3$

9. $\sqrt[3]{-8^2}$

6. e^0

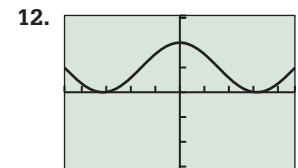
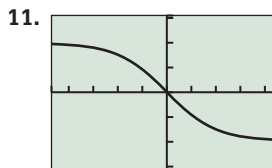
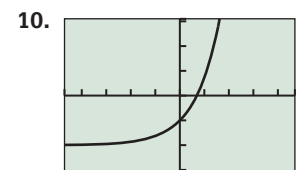
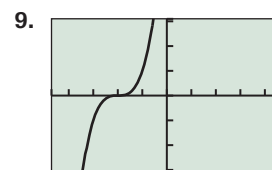
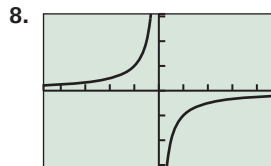
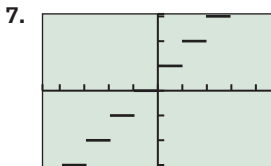
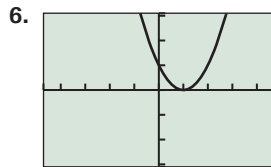
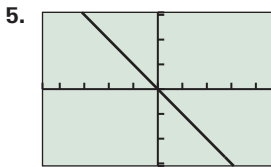
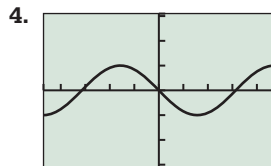
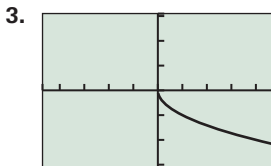
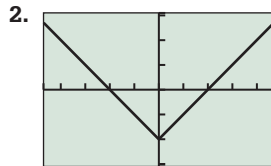
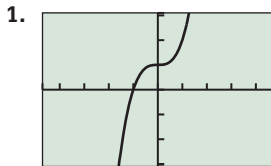
8. $\sqrt[3]{(-15)^3}$

10. $|1 - \pi| - \pi$

SECTION 1.3 EXERCISES

In Exercises 1–12, each graph is a slight variation on the graph of one of the twelve basic functions described in this section. Match the graph to one of the twelve functions (a)–(l) and then support your answer by checking the graph on your calculator. (All graphs are shown in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$.)

- (a) $y = -\sin x$ (b) $y = \cos x + 1$ (c) $y = e^x - 2$
 (d) $y = (x + 2)^3$ (e) $y = x^3 + 1$ (f) $y = (x - 1)^2$
 (g) $y = |x| - 2$ (h) $y = -1/x$ (i) $y = -x$
 (j) $y = -\sqrt{x}$ (k) $y = \text{int}(x + 1)$ (l) $y = 2 - 4/(1 + e^{-x})$



In Exercises 13–18, identify which of Exercises 1–12 display functions that fit the description given.

13. The function whose domain excludes zero.
 14. The function whose domain consists of all nonnegative real numbers.
 15. The two functions that have at least one point of discontinuity.
 16. The function that is not a *continuous function*.
 17. The six functions that are bounded below.
 18. The four functions that are bounded above.

In Exercises 19–28, identify which of the twelve basic functions fit the description given.

19. The four functions that are odd.
 20. The six functions that are increasing on their entire domains.
 21. The three functions that are decreasing on the interval $(-\infty, 0)$.
 22. The three functions with infinitely many local extrema.
 23. The three functions with no zeros.
 24. The three functions with range {all real numbers}.

25. The four functions that do *not* have end behavior $\lim_{x \rightarrow +\infty} f(x) = +\infty$.
26. The three functions with end behavior $\lim_{x \rightarrow -\infty} f(x) = -\infty$.
27. The four functions whose graphs look the same when turned upside-down and flipped about the y -axis.
28. The two functions whose graphs are identical except for a horizontal shift.

In Exercises 29–34, use your graphing calculator to produce a graph of the function. Then determine the domain and range of the function by looking at its graph.

29. $f(x) = x^2 - 5$ 30. $g(x) = |x - 4|$
 31. $h(x) = \ln(x + 6)$ 32. $k(x) = 1/x + 3$
 33. $s(x) = \text{int}(x/2)$ 34. $p(x) = (x + 3)^2$

In Exercises 35–42, graph the function. Then answer the following questions:

- (a) On what interval, if any, is the function increasing? Decreasing?
 (b) Is the function odd, even, or neither?
 (c) Give the function's extrema, if any.
 (d) How does the graph relate to a graph of one of the twelve basic functions?

35. $r(x) = \sqrt{x - 10}$ 36. $f(x) = \sin(x) + 5$
 37. $f(x) = 3/(1 + e^{-x})$ 38. $q(x) = e^x + 2$
 39. $h(x) = |x| - 10$ 40. $g(x) = 4 \cos(x)$
 41. $s(x) = |x - 2|$ 42. $f(x) = 5 - \text{abs}(x)$

43. Find the horizontal asymptotes for the graph shown in Exercise 11.
 44. Find the horizontal asymptotes for the graph of $f(x)$ in Exercise 37.

In Exercises 45–52, sketch the graph of the piecewise-defined function. (Try doing it without a calculator.) In each case, give any points of discontinuity.

45. $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$ 46. $g(x) = \begin{cases} x^3 & \text{if } x \leq 0 \\ e^x & \text{if } x > 0 \end{cases}$
 47. $h(x) = \begin{cases} |x| & \text{if } x < 0 \\ \sin x & \text{if } x \geq 0 \end{cases}$ 48. $w(x) = \begin{cases} 1/x & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$
 49. $f(x) = \begin{cases} \cos x & \text{if } x \leq 0 \\ e^x & \text{if } x > 0 \end{cases}$ 50. $g(x) = \begin{cases} |x| & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$
 51. $f(x) = \begin{cases} -3 - x & \text{if } x \leq 0 \\ 1 & \text{if } 0 < x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$
 52. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ |x| & \text{if } -1 \leq x < 1 \\ \text{int}(x) & \text{if } x \geq 1 \end{cases}$

53. **Writing to Learn** The function $f(x) = \sqrt{x^2}$ is one of our twelve basic functions written in another form.

- (a) Graph the function and identify which basic function it is.
 (b) Explain algebraically why the two functions are equal.

54. **Uncovering Hidden Behavior** The function $g(x) = \sqrt{x^2 + 0.0001} - 0.01$ is *not* one of our twelve basic functions written in another form.

- (a) Graph the function and identify which basic function it appears to be.
 (b) Verify numerically that it is not the basic function that it appears to be.

55. **Writing to Learn** The function $f(x) = \ln(e^x)$ is one of our twelve basic functions written in another form.

- (a) Graph the function and identify which basic function it is.
 (b) Explain how the equivalence of the two functions in (a) shows that the natural logarithm function is *not* bounded above (even though it *appears* to be bounded above in Figure 1.42).

56. **Writing to Learn** Let $f(x)$ be the function that gives the cost, in cents, to mail a letter that weighs x ounces. As of June 2002, the cost is 37 cents for a letter that weighs up to one ounce, plus 23 cents for each additional ounce or portion thereof.

- (a) Sketch a graph of $f(x)$.
 (b) How is this function similar to the greatest integer function? How is it different?

57. **Analyzing a Function** Set your calculator to DOT mode and graph the greatest integer function, $y = \text{int}(x)$, in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$. Then complete the following analysis.

BASIC FUNCTION

The Greatest Integer Function

$$f(x) = \text{int } x$$

Domain:

Range:

Continuity:

Increasing/decreasing behavior:

Symmetry:

Boundedness:

Local extrema:

Horizontal asymptotes:

Vertical asymptotes:

End behavior:

Standardized Test Questions

58. True or False The greatest integer function has an inverse function. Justify your answer.

59. True or False The logistic function has two horizontal asymptotes. Justify your answer.

In Exercises 60–63, you may use a graphing calculator to answer the question.

60. Multiple Choice Which function has range {all real numbers}?

- (A) $f(x) = 4 + \ln x$
- (B) $f(x) = 3 - 1/x$
- (C) $f(x) = 5/(1 + e^{-x})$
- (D) $f(x) = \text{int}(x - 2)$
- (E) $f(x) = 4 \cos x$

61. Multiple Choice Which function is bounded both above and below?

- (A) $f(x) = x^2 - 4$
- (B) $f(x) = (x - 3)^3$
- (C) $f(x) = 3e^x$
- (D) $f(x) = 3 + 1/(1 + e^{-x})$
- (E) $f(x) = 4 - |x|$

62. Multiple Choice Which of the following is the same as the restricted-domain function $f(x) = \text{int}(x)$, $0 \leq x < 2$?

$$(A) f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \\ 2 & \text{if } 1 < x < 2 \end{cases}$$

$$(B) f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \leq 1 \\ 2 & \text{if } 1 < x < 2 \end{cases}$$

$$(C) f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x < 2 \end{cases}$$

$$(D) f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 2 & \text{if } 1 \leq x < 2 \end{cases}$$

$$(E) f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 1 + x & \text{if } 1 \leq x < 2 \end{cases}$$

63. Multiple Choice Increasing Functions Which function is increasing on the interval $(-\infty, \infty)$?

- (A) $f(x) = \sqrt{3 + x}$
- (B) $f(x) = \text{int}(x)$
- (C) $f(x) = 2x^2$
- (D) $f(x) = \sin x$
- (E) $f(x) = 3/(1 + e^{-x})$

Explorations

64. Which is Bigger? For positive values of x , we wish to compare the values of the basic functions x^2 , x , and \sqrt{x} .

- (a) How would you order them from least to greatest?
- (b) Graph the three functions in the viewing window $[0, 30]$ by $[0, 20]$. Does the graph confirm your response in (a)?
- (c) Now graph the three functions in the viewing window $[0, 2]$ by $[0, 1.5]$.
- (d) Write a careful response to the question in (a) that accounts for all positive values of x .

65. Odds and Evens There are four odd functions and three even functions in the gallery of twelve basic functions. After multiplying these functions together pairwise in different combinations and exploring the graphs of the products, make a conjecture about the symmetry of:

- (a) a product of two odd functions.
- (b) a product of two even functions.
- (c) a product of an odd function and an even function.

66. Group Activity Assign to each student in the class the name of one of the twelve basic functions, but secretly so that no student knows the “name” of another. (The same function name could be given to several students, but all the functions should be used at least once.) Let each student make a one-sentence self-introduction to the class that reveals something personal “about who I am that really identifies me.” The rest of the students then write down their guess as to the function’s identity. Hints should be subtle and cleverly anthropomorphic. (For example, the absolute value function saying “I have a very sharp smile” is subtle and clever, while “I am absolutely valuable” is not very subtle at all.)

67. Pepperoni Pizzas For a statistics project, a student counted the number of pepperoni slices on pizzas of various sizes at a local pizzeria, compiling the following table:



Table 1.10

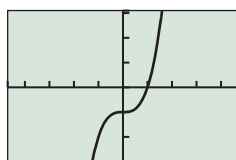
Type of Pizza	Radius	Pepperoni count
Personal	4"	12
Medium	6"	27
Large	7"	37
Extra Large	8"	48

- (a) Explain why the pepperoni count (P) ought to be proportional to the square of the radius (r).
- (b) Assuming that $P = k \cdot r^2$, use the data pair (4, 12) to find the value of k .
- (c) Does the algebraic model fit the rest of the data well?

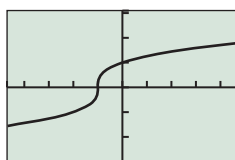
- (d) Some pizza places have charts showing their kitchen staff how much of each topping should be put on each size of pizza. Do you think this pizzeria uses such a chart? Explain.

Extending the Ideas

- 68. Inverse Functions** Two functions are said to be *inverses* of each other if the graph of one can be obtained from the graph of the other by reflecting it across the line $y = x$. For example, the functions with the graphs shown below are inverses of each other:



$[-4.7, 4.7]$ by $[-3.1, 3.1]$
(a)



$[-4.7, 4.7]$ by $[-3.1, 3.1]$
(b)

- (a) Two of the twelve basic functions in this section are inverses of each other. Which are they?

- (b) Two of the twelve basic functions in this section are their own inverses. Which are they?
- (c) If you restrict the domain of one of the twelve basic functions to $[0, \infty)$, it becomes the inverse of another one. Which are they?

69. Identifying a Function by Its Properties

- (a) Seven of the twelve basic functions have the property that $f(0) = 0$. Which five do not?
- (b) Only one of the twelve basic functions has the property that $f(x + y) = f(x) + f(y)$ for all x and y in its domain. Which one is it?
- (c) One of the twelve basic functions has the property that $f(x + y) = f(x)f(y)$ for all x and y in its domain. Which one is it?
- (d) One of the twelve basic functions has the property that $f(xy) = f(x) + f(y)$ for all x and y in its domain. Which one is it?
- (e) Four of the twelve basic functions have the property that $f(x) + f(-x) = 0$ for all x in their domains. Which four are they?

1.4

Building Functions from Functions

What you'll learn about

- Combining Functions Algebraically
- Composition of Functions
- Relations and Implicitly Defined Functions

... and why

Most of the functions that you will encounter in calculus and in real life can be created by combining or modifying other functions.

Combining Functions Algebraically

Knowing how a function is “put together” is an important first step when applying the tools of calculus. Functions have their own algebra based on the same operations we apply to real numbers (addition, subtraction, multiplication, and division). One way to build new functions is to apply these operations, using the following definitions.

DEFINITION Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with intersecting domains. Then for all values of x in the intersection, the algebraic combinations of f and g are defined by the following rules:

Sum: $(f + g)(x) = f(x) + g(x)$

Difference: $(f - g)(x) = f(x) - g(x)$

Product: $(fg)(x) = f(x)g(x)$

Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$

In each case, the domain of the new function consists of all numbers that belong to both the domain of f and the domain of g . As noted, the zeros of the denominator are excluded from the domain of the quotient.

Euler's function notation works so well in the above definitions that it almost obscures what is really going on. The “+” in the expression “ $(f + g)(x)$ ” stands for a brand new operation called *function addition*. It builds a new function, $f + g$, from the given functions f and g . Like any function, $f + g$ is defined by what it does: It takes a domain value x and returns a range value $f(x) + g(x)$. Note that the “+” sign in “ $f(x) + g(x)$ ” *does* stand for the familiar operation of real number addition. So, with the same symbol taking on different roles on either side of the equal sign, there is more to the above definitions than first meets the eye.

Fortunately, the definitions are easy to apply.

EXAMPLE 1 Defining New Functions Algebraically

Let $f(x) = x^2$ and $g(x) = \sqrt{x+1}$.

Find formulas for the functions $f+g$, $f-g$, fg , f/g , and gg . Give the domain of each.

SOLUTION We first determine that f has domain all real numbers and that g has domain $[-1, \infty)$. These domains overlap, the intersection being the interval $[-1, \infty)$. So:

$$(f+g)(x) = f(x) + g(x) = x^2 + \sqrt{x+1} \quad \text{with domain } [-1, \infty).$$

$$(f-g)(x) = f(x) - g(x) = x^2 - \sqrt{x+1} \quad \text{with domain } [-1, \infty).$$

$$(fg)(x) = f(x)g(x) = x^2\sqrt{x+1} \quad \text{with domain } [-1, \infty).$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{\sqrt{x+1}} \quad \text{with domain } (-1, \infty).$$

$$(gg)(x) = g(x)g(x) = (\sqrt{x+1})^2 \quad \text{with domain } [-1, \infty).$$

Note that we could express $(gg)(x)$ more simply as $x+1$. That would be fine, but the simplification would not change the fact that the domain of gg is (by definition) the interval $[-1, \infty)$. Under other circumstances the function $h(x) = x+1$ would have domain all real numbers, but under these circumstances it cannot; it is a product of two functions with restricted domains.

Now try Exercise 3.

Composition of Functions

It is not hard to see that the function $\sin(x^2)$ is built from the basic functions $\sin x$ and x^2 , but the functions are not put together by addition, subtraction, multiplication, or division. Instead, the two functions are combined by simply applying them in order—first the squaring function, then the sine function. This operation for combining functions, which has no counterpart in the algebra of real numbers, is called *function composition*.

DEFINITION Composition of Functions

Let f and g be two functions such that the domain of f intersects the range of g . The **composition f of g** , denoted $f \circ g$, is defined by the rule

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of all x -values in the domain of g that map to $g(x)$ -values in the domain of f . (See Figure 1.55.)

The composition g of f , denoted $g \circ f$, is defined similarly. In most cases $g \circ f$ and $f \circ g$ are different functions. (In the language of algebra, “function composition is not commutative.”)

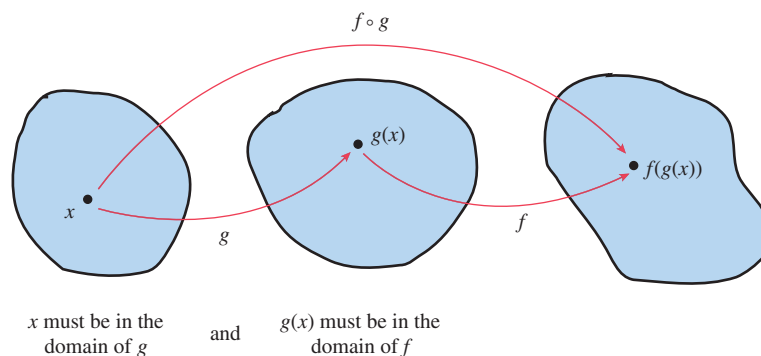


FIGURE 1.55 In the composition $f \circ g$, the function g is applied first and then f . This is the reverse of the order in which we read the symbols.

EXAMPLE 2 Composing Functions

Let $f(x) = e^x$ and $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and verify numerically that the functions $f \circ g$ and $g \circ f$ are not the same.

SOLUTION

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = e^{\sqrt{x}}$$

$$(g \circ f)(x) = g(f(x)) = g(e^x) = \sqrt{e^x}$$

One verification that these functions are not the same is that they have different domains: $f \circ g$ is defined only for $x \geq 0$, while $g \circ f$ is defined for all real numbers. We could also consider their graphs (Figure 1.56), which agree only at $x = 0$ and $x = 4$.

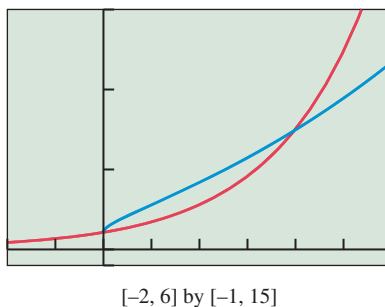


FIGURE 1.56 The graphs of $y = e^{\sqrt{x}}$ and $y = \sqrt{e^x}$ are not the same. (Example 2)

Finally, the graphs suggest a numerical verification: Find a single value of x for which $f(g(x))$ and $g(f(x))$ give different values. For example, $f(g(1)) = e$ and $g(f(1)) = \sqrt{e}$. The graph helps us to make a judicious choice of x . You do not want to check the functions at $x = 0$ and $x = 4$ and conclude that they are the same!

Now try Exercise 15.

EXPLORATION 1 Composition Calisthenics

One of the f functions in Column B can be composed with one of the g functions in Column C to yield each of the basic $f \circ g$ functions in Column A. Can you match the columns successfully without a graphing calculator? If you are having trouble, try it with a graphing calculator.

A	B	C
$f \circ g$	f	g
x	$x - 3$	$x^{0.6}$
x^2	$2x - 3$	x^2
$ x $	\sqrt{x}	$\frac{(x - 2)(x + 2)}{2}$
x^3	x^5	$\ln(e^3 x)$
$\ln x$	$ 2x + 4 $	$\frac{x}{2}$
$\sin x$	$1 - 2x^2$	$\frac{x + 3}{2}$
$\cos x$	$2 \sin x \cos x$	$\sin\left(\frac{x}{2}\right)$

CAUTION

We might choose to express $(f \circ g)$ more simply as $x - 1$. However, you must remember that the composition is restricted to the domain of $g(x) = \sqrt{x}$, or $[0, \infty]$. The domain of $x - 1$ is all real numbers. It is a good idea to work out the domain of a composition before you simplify the expression for $f(g(x))$. One way to simplify and maintain the restriction on the domain in Example 3 is to write $(f \circ g)(x) = x - 1, x \geq 0$.

EXAMPLE 3 Finding the Domain of a Composition

Let $f(x) = x^2 - 1$ and let $g(x) = \sqrt{x}$. Find the domains of the composite functions

- (a) $g \circ f$ (b) $f \circ g$

SOLUTION

- (a) We compose the functions in the order specified:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= \sqrt{x^2 - 1}\end{aligned}$$

For x to be in the domain of $g \circ f$, we must first find $f(x) = x^2 - 1$, which we can do for all real x . Then we must take the square root of the result, which we can only do for nonnegative values of $x^2 - 1$.

Therefore, the domain of $g \circ f$ consists of all real numbers for which $x^2 - 1 \geq 0$, namely the union $(-\infty, -1] \cup [1, \infty)$.

- (b) Again, we compose the functions in the order specified:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= (\sqrt{x})^2 - 1\end{aligned}$$

continued

For x to be in the domain of $f \circ g$, we must first be able to find $g(x) = \sqrt{x}$, which we can only do for nonnegative values of x . Then we must be able to square the result and subtract 1, which we can do for all real numbers.

Therefore, the domain of $f \circ g$ consists of the interval $[0, \infty)$.

Support Graphically

We can graph the composition functions to see if the grapher respects the domain restrictions. The screen to the left of each graph shows the set up in the “Y=” editor. Figure 1.57b shows the graph of $y = (g \circ f)(x)$, while Figure 1.57d shows the graph of $y = (f \circ g)(x)$. The graphs support our algebraic work quite nicely.

Now try Exercise 17.

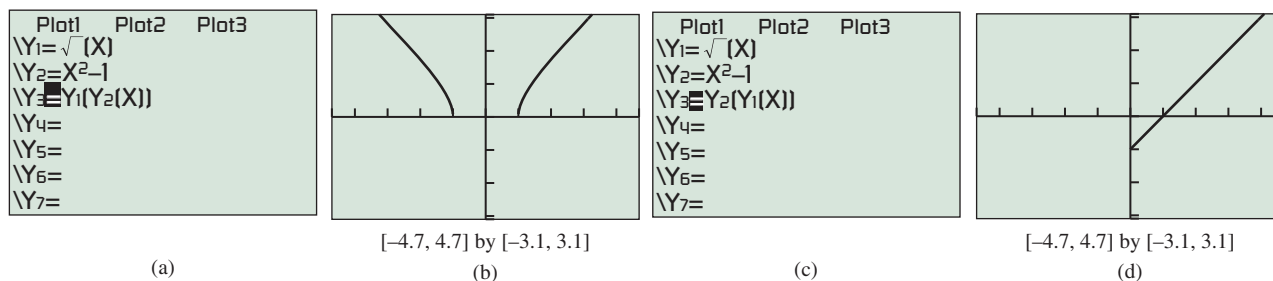


FIGURE 1.57 The functions Y_1 and Y_2 are composed to get the graphs of $y = (g \circ f)(x)$ and $y = (f \circ g)(x)$, respectively. The graphs support our conclusions about the domains of the two composite functions. (Example 3)

In Examples 2 and 3 two functions were *composed* to form new functions. There are times in calculus when we need to reverse the process. That is, we may begin with a function h and *decompose* it by finding functions whose composition is h .



EXAMPLE 4 Decomposing Functions

For each function h , find functions f and g such that $h(x) = f(g(x))$.

(a) $h(x) = (x + 1)^2 - 3(x + 1) + 4$

(b) $h(x) = \sqrt{x^3 + 1}$

SOLUTION

(a) We can see that h is quadratic in $x + 1$. Let $f(x) = x^2 - 3x + 4$ and let $g(x) = x + 1$. Then

$$h(x) = f(g(x)) = f(x + 1) = (x + 1)^2 - 3(x + 1) + 4.$$

(b) We can see that h is the square root of the function $x^3 + 1$. Let $f(x) = \sqrt{x}$ and let $g(x) = x^3 + 1$. Then

$$h(x) = f(g(x)) = f(x^3 + 1) = \sqrt{x^3 + 1}.$$

Now try Exercise 25.

There is often more than one way to decompose a function. For example, an alternate way to decompose $h(x) = \sqrt{x^3 + 1}$ in Example 4b is to let $f(x) = \sqrt{x + 1}$ and let $g(x) = x^3$. Then $h(x) = f(g(x)) = f(x^3) = \sqrt{x^3 + 1}$.

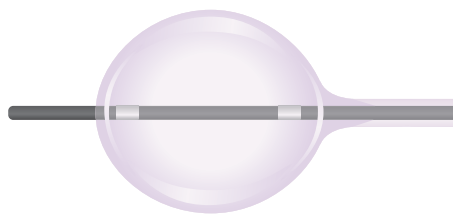
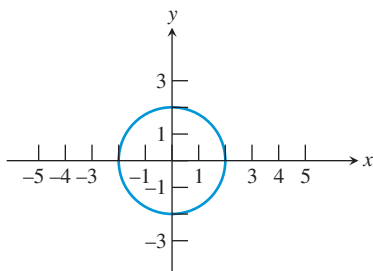


FIGURE 1.58 (Example 5)

FIGURE 1.59 A circle of radius 2 centered at the origin. This set of ordered pairs (x, y) defines a *relation* that is not a function, because the graph fails the vertical line test.

GRAPHING RELATIONS

Relations that are not functions are often not easy to graph. We will study some special cases later in the course (circles, ellipses, etc.), but some simple-looking relations like Example 6 are difficult to graph. Nor do our calculators help much, because the equation can not be put into “Y1=” form. Interestingly, we do know that the graph of the relation in Example 6, whatever it looks like, fails the vertical line test.

EXAMPLE 5 Modeling with Function Composition

In the medical procedure known as angioplasty, doctors insert a catheter into a heart vein (through a large peripheral vein) and inflate a small, spherical balloon on the tip of the catheter. Suppose the balloon is inflated at a constant rate of 44 cubic millimeters per second. (See Figure 1.58.)

- Find the volume after t seconds
- When the volume is V , what is the radius r ?
- Write an equation that gives the radius r as a function of the time. What is the radius after 5 seconds?

SOLUTION

(a) After t seconds, the volume will be $44t$.

(b) **Solve Algebraically**

$$\begin{aligned}\frac{4}{3}\pi r^3 &= v \\ r^3 &= \frac{3v}{4\pi} \\ r &= \sqrt[3]{\frac{3v}{4\pi}}\end{aligned}$$

(c) Substituting $44t$ for V gives $r = \sqrt[3]{\frac{3 \cdot 44t}{4\pi}}$ or $r = \sqrt[3]{\frac{33t}{\pi}}$. After 5 seconds, the radius will be $r = \sqrt[3]{\frac{33 \cdot 5}{\pi}} \approx 3.74$ mm.

Now try Exercise 31.

Relations and Implicitly Defined Functions

There are many useful curves in mathematics that fail the vertical line test and therefore are not graphs of functions. One such curve is the circle in Figure 1.59. While y is not related to x as a function in this instance, there is certainly some sort of relationship going on. In fact, not only does the shape of the graph show a significant *geometric* relationship among the points, but the ordered pairs (x, y) exhibit a significant *algebraic* relationship as well: They consist exactly of the solutions to the equation $x^2 + y^2 = 4$.

The general term for a set of ordered pairs (x, y) is a **relation**. If the relation happens to relate a *single* value of y to each value of x , then the relation is also a function and its graph will pass the vertical line test. In the case of the circle with equation $x^2 + y^2 = 4$, both $(0, 2)$ and $(0, -2)$ are in the relation, so y is not a function of x .

EXAMPLE 6 Verifying Pairs in a Relation

Determine which of the ordered pairs $(2, -5)$, $(1, 3)$, and $(2, 1)$ are in the relation defined by $x^2y + y^2 = 5$. Is the relation a function?

SOLUTION We simply substitute the x - and y -coordinates of the ordered pairs into $x^2y + y^2$ and see if we get 5.

continued

$$(2, -5): \quad (2)^2(-5) + (-5)^2 = 5 \quad \text{Substitute } x = 2, y = -5.$$

$$(1, 3): \quad (1)^2(3) + (3)^2 = 12 \neq 5 \quad \text{Substitute } x = 1, y = 3.$$

$$(2, 1): \quad (2)^2(1) + (1)^2 = 5 \quad \text{Substitute } x = 2, y = 1.$$

So, $(2, -5)$ and $(2, 1)$ are in the relation, but $(1, 3)$ is not.

Since the equation relates two different y -values (-5 and 1) to the same x -value (2), the relation cannot be a function.

Now try Exercise 35.

Let us revisit the circle $x^2 + y^2 = 4$. While it is not a function itself, we can split it into two equations that *do* define functions, as follows:

$$x^2 + y^2 = 4$$

$$y^2 = 4 - x^2$$

$$y = +\sqrt{4 - x^2} \text{ or } y = -\sqrt{4 - x^2}$$

The graphs of these two functions are, respectively, the upper and lower semicircles of the circle in Figure 1.59. They are shown in Figure 1.60. Since all the ordered pairs in either of these functions satisfy the equation $x^2 + y^2 = 4$, we say that the relation given by the equation defines the two functions **implicitly**.



EXAMPLE 7 Using Implicitly Defined Functions

Describe the graph of the relation $x^2 + 2xy + y^2 = 1$.

SOLUTION This looks like a difficult task at first, but notice that the expression on the left of the equal sign is a factorable trinomial. This enables us to split the relation into two implicitly defined functions as follows:

$$x^2 + 2xy + y^2 = 1$$

$$(x + y)^2 = 1 \quad \text{Factor.}$$

$$x + y = \pm 1 \quad \text{Extract square roots.}$$

$$x + y = 1 \text{ or } x + y = -1$$

$$y = -x + 1 \text{ or } y = -x - 1 \quad \text{Solve for } y.$$

The graph consists of two parallel lines (Figure 1.61), each the graph of one of the implicitly defined functions.

Now try Exercise 37.

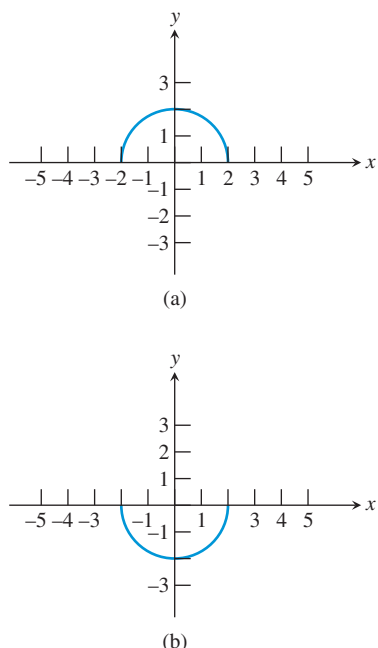


FIGURE 1.60 The graphs of
(a) $y = +\sqrt{4 - x^2}$ and
(b) $y = -\sqrt{4 - x^2}$. In each case, y is defined as a function of x . These two functions are defined *implicitly* by the relation $x^2 + y^2 = 4$.

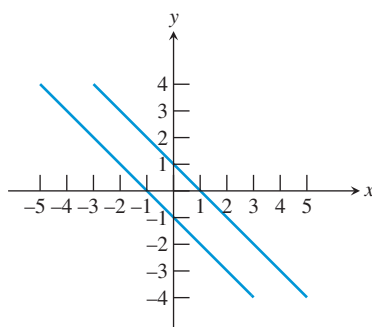


FIGURE 1.61 The graph of the relation $x^2 + 2xy + y^2 = 1$ (Example 7)

QUICK REVIEW 1.4 (For help, go to Sections P.1, 1.2, and 1.3.)

In Exercises 1–10, find the domain of the function and express it in interval notation.

$$1. f(x) = \frac{x-2}{x+3}$$

$$2. g(x) = \ln(x-1)$$

$$3. f(t) = \sqrt{5-t}$$

$$4. g(x) = \frac{3}{\sqrt{2x-1}}$$

$$5. f(x) = \sqrt{\ln(x)}$$

$$6. h(x) = \sqrt{1-x^2}$$

$$7. f(t) = \frac{t+5}{t^2+1}$$

$$8. g(t) = \ln(|t|)$$

$$9. f(x) = \frac{1}{\sqrt{1-x^2}}$$

$$10. g(x) = 2$$

SECTION 1.4 EXERCISES

In Exercises 1–4, find formulas for the functions $f+g$, $f-g$, and fg . Give the domain of each.

$$1. f(x) = 2x-1; g(x) = x^2 \quad 2. f(x) = (x-1)^2; g(x) = 3-x$$

$$3. f(x) = \sqrt{x}; g(x) = \sin x \quad 4. f(x) = \sqrt{x+5}; g(x) = |x+3|$$

In Exercises 5–8, find formulas for f/g and g/f . Give the domain of each.

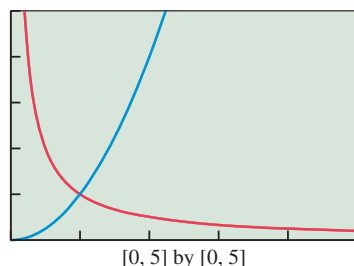
$$5. f(x) = \sqrt{x+3}; g(x) = x^2$$

$$6. f(x) = \sqrt{x-2}; g(x) = \sqrt{x+4}$$

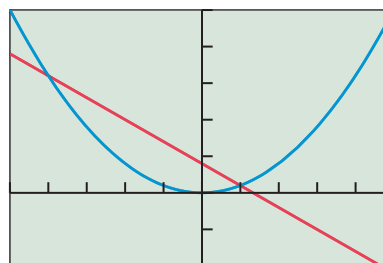
$$7. f(x) = x^2; g(x) = \sqrt{1-x^2}$$

$$8. f(x) = x^3; g(x) = \sqrt[3]{1-x^3}$$

9. $f(x) = x^2$ and $g(x) = 1/x$ are shown below in the viewing window $[0, 5]$ by $[0, 5]$. Sketch the graph of the sum $(f+g)(x)$ by adding the y -coordinates directly from the graphs. Then graph the sum on your calculator and see how close you came.



10. The graphs of $f(x) = x^2$ and $g(x) = 4-3x$ are shown in the viewing window $[-5, 5]$ by $[-10, 25]$. Sketch the graph of the difference $(f-g)(x)$ by subtracting the y -coordinates directly from the graphs. Then graph the difference on your calculator and see how close you came.



$[-5, 5]$ by $[-10, 25]$

In Exercises 11–14, find $(f \circ g)(3)$ and $(g \circ f)(-2)$.

$$11. f(x) = 2x-3; g(x) = x+1$$

$$12. f(x) = x^2-1; g(x) = 2x-3$$

$$13. f(x) = x^2+4; g(x) = \sqrt{x+1}$$

$$14. f(x) = \frac{x}{x+1}; g(x) = 9-x^2$$

In Exercises 15–22, find $f(g(x))$ and $g(f(x))$. State the domain of each.

$$15. f(x) = 3x+2; g(x) = x-1$$

$$16. f(x) = x^2-1; g(x) = \frac{1}{x-1}$$

$$17. f(x) = x^2-2; g(x) = \sqrt{x+1}$$

$$18. f(x) = \frac{1}{x-1}; g(x) = \sqrt{x}$$

$$19. f(x) = x^2; g(x) = \sqrt{1-x^2}$$

$$20. f(x) = x^3; g(x) = \sqrt[3]{1-x^3}$$

$$21. f(x) = \frac{1}{2x}; g(x) = \frac{1}{3x}$$

$$22. f(x) = \frac{1}{x+1}; g(x) = \frac{1}{x-1}$$

In Exercises 23–30, find $f(x)$ and $g(x)$ so that the function can be described as $y = f(g(x))$. (There may be more than one possible decomposition.)

23. $y = \sqrt{x^2 - 5x}$

24. $y = (x^3 + 1)^2$

25. $y = |3x - 2|$

26. $y = \frac{1}{x^3 - 5x + 3}$

27. $y = (x - 3)^5 + 2$

28. $y = e^{\sin x}$

29. $y = \cos(\sqrt{x})$

30. $y = (\tan x)^2 + 1$

- 31. Weather Balloons** A high-altitude spherical weather balloon expands as it rises due to the drop in atmospheric pressure. Suppose that the radius r increases at the rate of 0.03 inches per second and that $r = 48$ inches at time $t = 0$. Determine an equation that models the volume V of the balloon at time t and find the volume when $t = 300$ seconds.



- 32. A Snowball's Chance** Jake stores a small cache of 4-inch diameter snowballs in the basement freezer, unaware that the freezer's self-defrosting feature will cause each snowball to lose about 1 cubic inch of volume every 40 days. He remembers them a year later (call it 360 days) and goes to retrieve them. What is their diameter then?
- 33. Satellite Photography** A satellite camera takes a rectangular-shaped picture. The smallest region that can be photographed is a 5-km by 7-km rectangle. As the camera zooms out, the length l and width w of the rectangle increase at a rate of 2 km/sec. How long does it take for the area A to be at least 5 times its original size?
- 34. Computer Imaging** New Age Special Effects, Inc., prepares computer software based on specifications prepared by film directors. To simulate an approaching vehicle, they begin with a computer image of a 5-cm by 7-cm by 3-cm box. The program increases each dimension at a rate of 2 cm/sec. How long does it take for the volume V of the box to be at least 5 times its initial size?
- 35.** Which of the ordered pairs $(1, 1)$, $(4, -2)$, and $(3, -1)$ are in the relation given by $3x + 4y = 5$?
- 36.** Which of the ordered pairs $(5, 1)$, $(3, 4)$, and $(0, -5)$ are in the relation given by $x^2 + y^2 = 25$?

In Exercises 37–44, find two functions defined implicitly by the given relation.

37. $x^2 + y^2 = 25$

38. $x + y^2 = 25$

39. $x^2 - y^2 = 25$

40. $3x^2 - y^2 = 25$

41. $x + |y| = 1$

42. $x - |y| = 1$

43. $y^2 = x^2$

44. $y^2 = x$

Standardized Test Questions

- 45. True or False** The domain of the quotient function $(f/g)(x)$ consists of all numbers that belong to both the domain of f and the domain of g . Justify your answer.
- 46. True or False** The domain of the product function $(fg)(x)$ consists of all numbers that belong to either the domain of f or the domain of g . Justify your answer.

You may use a graphing calculator when solving Exercises 47–50.

- 47. Multiple Choice** Suppose f and g are functions with domain all real numbers. Which of the following statements is *not* necessarily true?
- (A) $(f + g)(x) = (g + f)(x)$ (B) $(fg)(x) = (gf)(x)$
 (C) $f(g(x)) = g(f(x))$ (D) $(f - g)(x) = -(g - f)(x)$
 (E) $(f \circ g)(x) = f(g(x))$
- 48. Multiple Choice** If $f(x) = x - 7$ and $g(x) = \sqrt{4 - x}$, what is the domain of the function fg ?
- (A) $(-\infty, 4)$ (B) $(-\infty, 4]$ (C) $(4, \infty)$
 (D) $[4, \infty)$ (E) $(4, 7) \cup (7, \infty)$
- 49. Multiple Choice** If $f(x) = x^2 + 1$, then $(f \circ f)(x) =$
- (A) $2x^2 + 2$ (B) $2x^2 + 1$ (C) $x^4 + 1$
 (D) $x^4 + 2x^2 + 1$ (E) $x^4 + 2x^2 + 2$
- 50. Multiple Choice** Which of the following relations defines the function $y = |x|$ implicitly?
- (A) $y = x$ (B) $y^2 = x^2$ (C) $y^3 = x^3$
 (D) $x^2 + y^2 = 1$ (E) $x = |y|$

Explorations

- 51. Three on a Match** Match each function f with a function g and a domain D so that $(f \circ g)(x) = x^2$ with domain D .

f	g	D
e^x	$\sqrt{2 - x}$	$x \neq 0$
$(x^2 + 2)^2$	$x + 1$	$x \neq 1$
$(x^2 - 2)^2$	$2 \ln x$	$(0, \infty)$
$\frac{1}{(x - 1)^2}$	$\frac{1}{x - 1}$	$[2, \infty)$
$x^2 - 2x + 1$	$\sqrt{x - 2}$	$(-\infty, 2]$
$\left(\frac{x + 1}{x}\right)^2$	$\frac{x + 1}{x}$	$(-\infty, \infty)$

52. Be a g Whiz Let $f(x) = x^2 + 1$. Find a function g so that

- (a) $(fg)(x) = x^4 - 1$
- (b) $(f + g)(x) = 3x^2$
- (c) $(f/g)(x) = 1$
- (d) $f(g(x)) = 9x^4 + 1$
- (e) $g(f(x)) = 9x^4 + 1$

Extending the Ideas

53. Identifying Identities An *identity* for a function operation is a function that combines with a given function f to return the same function f . Find the identity functions for the following operations:

(a) Function addition. That is, find a function g such that $(f + g)(x) = (g + f)(x) = f(x)$.

(b) Function multiplication. That is, find a function g such that $(fg)(x) = (gf)(x) = f(x)$.

(c) Function composition. That is, find a function g such that $(f \circ g)(x) = (g \circ f)(x) = f(x)$.

54. Is Function Composition Associative? You already know that function composition is not commutative; that is, $(f \circ g)(x) \neq (g \circ f)(x)$. But is function composition associative? That is, does $(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x)$? Explain your answer.

55. Revisiting Example 6 Solve $x^2y + y^2 = 5$ for y using the quadratic formula and graph the pair of implicit functions.

1.5

Parametric Relations and Inverses

What you'll learn about

- Defined Relations Parametrically
- Inverse Relations and Inverse Functions

... and why

Some functions and graphs can best be defined parametrically, while some others can be best understood as inverses of functions we already know.

Relations Defined Parametrically

Another natural way to define functions or, more generally, relations, is to define *both* elements of the ordered pair (x, y) in terms of another variable t , called a **parameter**. We illustrate with an example.

EXAMPLE 1 Defining a Function Parametrically

Consider the set of all ordered pairs (x, y) defined by the equations

$$x = t + 1$$

$$y = t^2 + 2t$$

where t is any real number.

- Find the points determined by $t = -3, -2, -1, 0, 1, 2$, and 3 .
- Find an algebraic relationship between x and y . (This is often called “eliminating the parameter.”) Is y a function of x ?
- Graph the relation in the (x, y) plane.

SOLUTION

- Substitute each value of t into the formulas for x and y to find the point that it determines parametrically:

t	$x = t + 1$	$y = t^2 + 2t$	(x, y)
-3	-2	3	$(-2, 3)$
-2	-1	0	$(-1, 0)$
-1	0	-1	$(0, -1)$
0	1	0	$(1, 0)$
1	2	3	$(2, 3)$
2	3	8	$(3, 8)$
3	4	15	$(4, 15)$

- We can find the relationship between x and y algebraically by the method of substitution. First solve for t in terms of x to obtain $t = x - 1$.

$$y = t^2 + 2t \quad \text{Given}$$

$$y = (x - 1)^2 + 2(x - 1) \quad t = x - 1$$

$$= x^2 - 2x + 1 + 2x - 2 \quad \text{Expand.}$$

$$= x^2 - 1 \quad \text{Simplify.}$$

This is consistent with the ordered pairs we had found in the table. As t varies over all real numbers, we will get all the ordered pairs in the relation $y = x^2 - 1$, which does indeed define y as a function of x .

continued

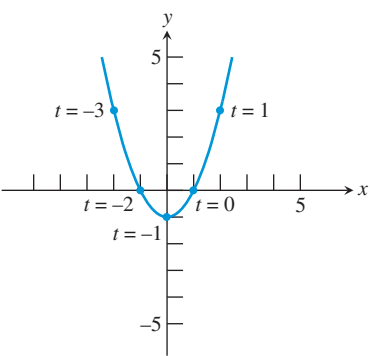


FIGURE 1.62 (Example 1)

(c) Since the parametrically defined relation consists of all ordered pairs in the relation $y = x^2 - 1$, we can get the graph by simply graphing the parabola $y = x^2 - 1$. See Figure 1.62. **Now try Exercise 5.**

EXAMPLE 2 Using a Graphing Calculator in Parametric Mode

Consider the set of all ordered pairs (x, y) defined by the equations

$$\begin{aligned}x &= t^2 + 2t \\ y &= t + 1\end{aligned}$$

where t is any real number.

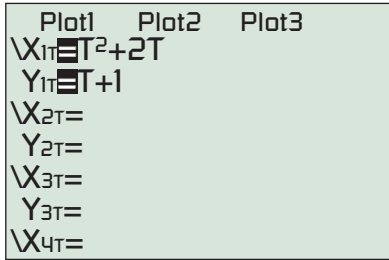
- (a) Use a graphing calculator to find the points determined by $t = -3, 2, -1, 0, 1, 2$, and 3 .
- (b) Use a graphing calculator to graph the relation in the (x, y) plane.
- (c) Is y a function of x ?
- (d) Find an algebraic relationship between x and y .

t	(x, y)
-3	(3, -2)
-2	(0, -1)
-1	(-1, 0)
0	(0, 1)
1	(3, 2)
2	(8, 3)
3	(15, 4)

SOLUTION

- (a) When the calculator is in *parametric mode*, the “Y =” screen provides a space to enter both X and Y as functions of the parameter T (Figure 1.63a). After entering the functions, use the table setup in Figure 1.63b to obtain the table shown in Figure 1.63c. The table shows, for example, that when $T = -3$ we have $X1T = 3$ and $Y1T = -2$, so the ordered pair corresponding to $t = -3$ is $(3, -2)$.
- (b) In parametric mode, the “WINDOW” screen contains the usual x -axis information, as well as “Tmin,” “Tmax,” and “Tstep” (Figure 1.64a). To include most of the points listed in part (a), we set $Xmin = -5$, $Xmax = 5$, $Ymin = -3$, and $Ymax = 3$. Since $t = y - 1$, we set $Tmin$ and $Tmax$ to values one less than those for $Ymin$ and $Ymax$.

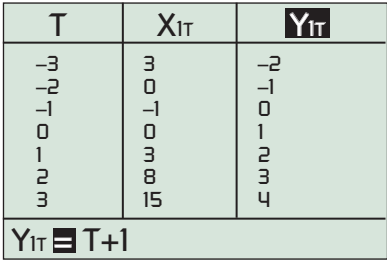
continued



(a)



(b)



(c)

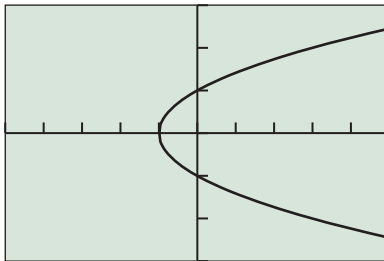
FIGURE 1.63 Using the table feature of a grapher set in parametric mode. (Example 2)

```

WINDOW
Tmin=-4
Tmax=2
Tstep=.1
Xmin=-5
Xmax=5
Xscl=1
↓Ymin=-3

```

(a)



[-5, 5] by [-3, 3]

(b)

FIGURE 1.64 The graph of a parabola in parametric mode on a graphing calculator. (Example 2)

TIME FOR T

Functions defined by parametric equations are frequently encountered in problems of motion, where the x - and y -coordinates of a moving object are computed as functions of time. This makes time the parameter, which is why you almost always see parameters given as “ t ” in parametric equations.

The value of T_{step} determines how far the grapher will go from one value of t to the next as it computes the ordered pairs. With $T_{\text{max}} - T_{\text{min}} = 6$ and $T_{\text{step}} = 0.1$, the grapher will compute 60 points, which is sufficient. (The more points, the smoother the graph. See Exploration 1.) The graph is shown in Figure 1.64b. Use trace to find some of the points found in (a).

- (c) No, y is not a function of x . We can see this from part (a) because $(0, -1)$ and $(0, 1)$ have the same x -value but different y -values. Alternatively, notice that the graph in (b) fails the vertical line test.
- (d) We can use the same algebraic steps as in Example 1 to get the relation in terms of x and y : $x = y^2 - 1$. **Now try Exercise 7.**

EXPLORATION 1 Watching your Tstep

- Graph the parabola in Example 2 in parametric mode as described in the solution. Press TRACE and observe the values of T , X , and Y . At what value of T does the calculator begin tracing? What point on the parabola results? (It's off the screen.) At what value of T does it stop tracing? What point on the parabola results? How many points are computed as you trace from start to finish?
- Leave everything else the same and change the T_{step} to 0.01. Do you get a smoother graph? Why or why not?
- Leave everything else the same and change the T_{step} to 1. Do you get a smoother graph? Why or why not?
- What effect does the T_{step} have on the speed of the grapher? Is this easily explained?
- Now change the T_{step} to 2. Why does the left portion of the parabola disappear? (It may help to TRACE along the curve.)
- Change the T_{step} back to 0.1 and change the T_{min} to -1 . Why does the bottom side of the parabola disappear? (Again, it may help to TRACE.)
- Make a change to the window that will cause the grapher to show the bottom side of the parabola but not the top.

Inverse Relations and Inverse Functions

What happens when we reverse the coordinates of all the ordered pairs in a relation? We obviously get another relation, as it is another set of ordered pairs, but does it bear any resemblance to the original relation? If the original relation happens to be a function, will the new relation also be a function?

We can get some idea of what happens by examining Examples 1 and 2. The ordered pairs in Example 2 can be obtained by simply reversing the coordinates of the ordered pairs in Example 1. This is because we set up Example 2 by switching the parametric equations for x and y that we used in Example 1. We say that the relation in Example 2 is the *inverse relation* of the relation in Example 1.

DEFINITION Inverse Relation

The ordered pair (a, b) is in a relation if and only if the ordered pair (b, a) is in the **inverse relation**.

We will study the connection between a relation and its inverse. We will be most interested in inverse relations that happen to be *functions*. Notice that the graph of the inverse relation in Example 2 fails the vertical line test and is therefore not the graph of a function. Can we predict this failure by considering the graph of the original relation in Example 1? Figure 1.65 suggests that we can.

The inverse graph in Figure 1.65b fails the vertical line test because two different y -values have been paired with the same x -value. This is a direct consequence of the fact that the original relation in Figure 1.65a paired two different x -values with the same y -value. The inverse graph fails the *vertical* line test precisely because the original graph fails the *horizontal* line test. This gives us a test for relations whose inverses are functions.

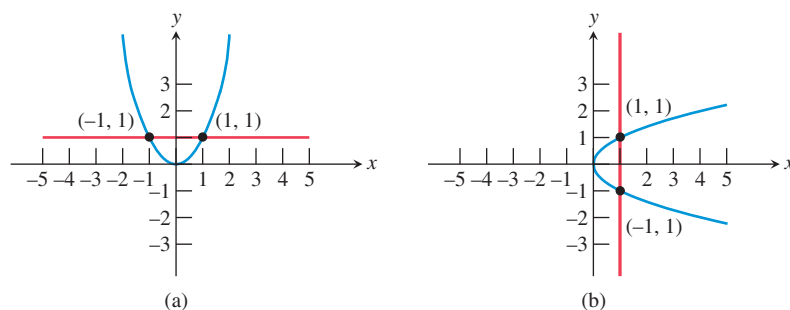


FIGURE 1.65 The inverse relation in (b) fails the vertical line test because the original relation in (a) fails the horizontal line test.

Horizontal Line Test

The inverse of a relation is a function if and only if each horizontal line intersects the graph of the original relation in at most one point.

EXAMPLE 3 Applying the Horizontal Line Test

Which of the graphs (1)–(4) in Figure 1.66 are graphs of

- (a) relations that are functions?
- (b) relations that have inverses that are functions?

SOLUTION

- (a) Graphs (1) and (4) are graphs of functions because these graphs pass the vertical line test. Graphs (2) and (3) are not graphs of functions because these graphs fail the vertical line test.

continued

(b) Graphs (1) and (2) are graphs of relations whose inverses are functions because these graphs pass the horizontal line test. Graphs (3) and (4) fail the horizontal line test so their inverse relations are not functions.

Now try Exercise 9.

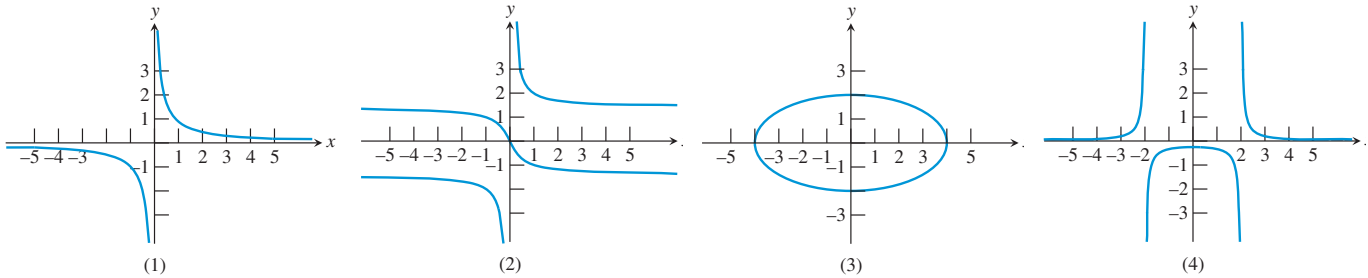


FIGURE 1.66 (Example 3)

A function whose inverse is a function has a graph that passes both the horizontal and vertical line tests (such as graph (1) in Example 3). Such a function is **one-to-one**, since every x is paired with a unique y and every y is paired with a unique x .

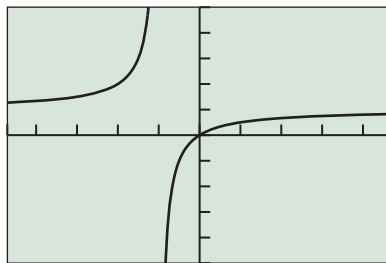
CAUTION ABOUT FUNCTION NOTATION

The symbol f^{-1} is read “ f inverse” and should never be confused with the reciprocal of f . If f is a function, the symbol f^{-1} , can only mean f inverse. The reciprocal of f must be written as $1/f$.

DEFINITION Inverse Function

If f is a one-to-one function with domain D and range R , then the **inverse function of f** , denoted f^{-1} , is the function with domain R and range D defined by

$$f^{-1}(b) = a \quad \text{if and only if} \quad f(a) = b.$$



$[-4.7, 4.7]$ by $[-5, 5]$

FIGURE 1.67 The graph of $f(x) = x/(x+1)$. (Example 4)

EXAMPLE 4 Finding an Inverse Function Algebraically

Find an equation for $f^{-1}(x)$ if $f(x) = x/(x+1)$.

SOLUTION The graph of f in Figure 1.67 suggests that f is one-to-one. The original function satisfies the equation $y = x/(x+1)$. If f truly is one-to-one, the inverse function f^{-1} will satisfy the equation $x = y/(y+1)$. (Note that we just switch the x and the y .)

If we solve this new equation for y we will have a formula for $f^{-1}(x)$:

$$\begin{aligned} x &= \frac{y}{y+1} && \text{Multiply by } y+1. \\ x(y+1) &= y && \text{Distributive property} \\ xy+x &= y && \\ xy-y &= -x && \text{Isolate the } y \text{ terms.} \\ y(x-1) &= -x && \text{Factor out } y. \\ y &= \frac{-x}{x-1} && \text{Divide by } x-1. \\ y &= \frac{x}{1-x} && \text{Multiply numerator and denominator by } -1. \end{aligned}$$

Therefore $f^{-1}(x) = x/(1-x)$.

Now try Exercise 15.

Let us candidly admit two things regarding Example 4 before moving on to a graphical model for finding inverses. First, many functions are not one-to-one and so do not have inverse functions. Second, the algebra involved in finding an inverse function in the manner of Example 4 can be extremely difficult. We will actually find very few inverses this way. As you will learn in future chapters, we will usually rely on our understanding of how f maps x to y to understand how f^{-1} maps y to x .

It is possible to use the graph of f to produce a graph of f^{-1} without doing any algebra at all, thanks to the following geometric reflection property:

The Inverse Reflection Principle

The points (a, b) and (b, a) in the coordinate plane are symmetric with respect to the line $y = x$. The points (a, b) and (b, a) are **reflections** of each other across the line $y = x$.

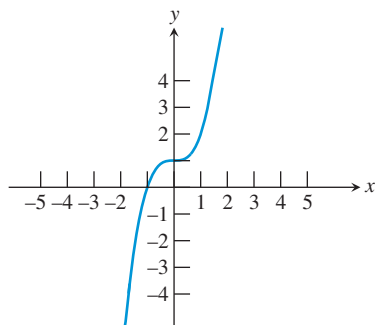


FIGURE 1.68 The graph of a one-to-one function. (Example 5)



EXAMPLE 5 Finding an Inverse Function Graphically

The graph of a function $y = f(x)$ is shown in Figure 1.68. Sketch a graph of the function $y = f^{-1}(x)$. Is f a one-to-one function?

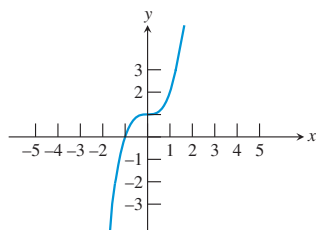
SOLUTION We need not find a formula for $f^{-1}(x)$. All we need to do is to find the reflection of the given graph across the line $y = x$. This can be done geometrically.

Imagine a mirror along the line $y = x$ and draw the reflection of the given graph in the mirror. (See Figure 1.69.)

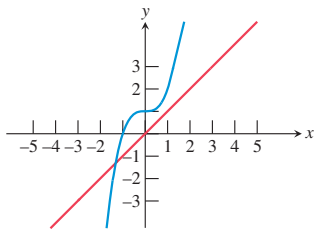
Another way to visualize this process is to imagine the graph to be drawn on a large pane of glass. Imagine the glass rotating around the line $y = x$ so that the *positive* x -axis switches places with the *positive* y -axis. (The back of the glass must be rotated to the front for this to occur.) The graph of f will then become the graph of f^{-1} .

Since the inverse of f has a graph that passes the horizontal and vertical line test, f is a one-to-one function.

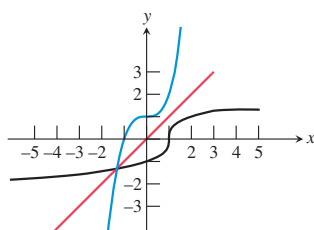
Now try Exercise 23.



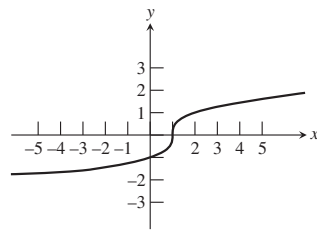
The graph of f .



The mirror $y = x$.



The reflection.



The graph of f^{-1} .

FIGURE 1.69 The Mirror Method. The graph of f is reflected in an imaginary mirror along the line $y = x$ to produce the graph of f^{-1} . (Example 5)

There is a natural connection between inverses and function composition that gives further insight into what an inverse actually does: It “undoes” the action of the original function. This leads to the following rule:

The Inverse Composition Rule

A function f is one-to-one with inverse function g if and only if

$$f(g(x)) = x \text{ for every } x \text{ in the domain of } g, \text{ and}$$

$$g(f(x)) = x \text{ for every } x \text{ in the domain of } f.$$

EXAMPLE 6 Verifying Inverse Functions

Show algebraically that $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$ are inverse functions.

SOLUTION We use the Inverse Composition Rule.

$$f(g(x)) = f(\sqrt[3]{x-1}) = (\sqrt[3]{x-1})^3 + 1 = x - 1 + 1 = x$$

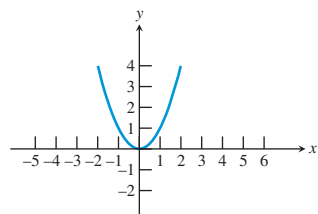
$$g(f(x)) = g(x^3 + 1) = \sqrt[3]{(x^3 + 1) - 1} = \sqrt[3]{x^3} = x$$

Since these equations are true for all x , the Inverse Composition Rule guarantees that f and g are inverses.

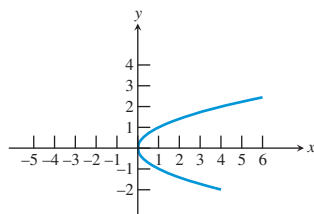
You do not have far to go to find graphical support of this algebraic verification, since these are the functions whose graphs are shown in Example 5!

Now try Exercise 27.

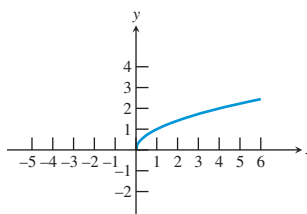
Some functions are so important that we need to study their inverses even though they are not one-to-one. A good example is the square root function, which is the “inverse” of the square function. It is not the inverse of the *entire* squaring function, because the full parabola fails the horizontal line test. Figure 1.70 shows that the function $y = \sqrt{x}$ is really the inverse of a “restricted-domain” version of $y = x^2$ defined only for $x \geq 0$.



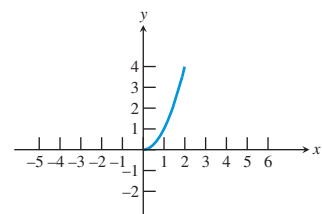
The graph of $y = x^2$ (not one-to-one).



The inverse relation of $y = x^2$ (not a function).



The graph of $y = \sqrt{x}$ (a function).



The graph of the function whose inverse is $y = \sqrt{x}$.

FIGURE 1.70 The function $y = x^2$ has no inverse function, but $y = \sqrt{x}$ is the inverse function of $y = x^2$ on the restricted domain $[0, \infty)$.

The consideration of domains adds a refinement to the algebraic inverse-finding method of Example 4, which we now summarize:

How to Find an Inverse Function Algebraically

Given a formula for a function f , proceed as follows to find a formula for f^{-1} .

1. Determine that there is a function f^{-1} by checking that f is one-to-one. State any restrictions on the domain of f . (Note that it might be necessary to impose some to get a one-to-one version of f .)
2. Switch x and y in the formula $y = f(x)$.
3. Solve for y to get the formula $y = f^{-1}(x)$. State any restrictions on the domain of f^{-1} .

EXAMPLE 7 Finding an Inverse Function

Show that $f(x) = \sqrt{x+3}$ has an inverse function and find a rule for $f^{-1}(x)$. State any restrictions on the domains of f and f^{-1} .

SOLUTION

Solve Algebraically

The graph of f passes the horizontal line test, so f has an inverse function (Figure 1.71). Note that f has domain $[-3, \infty)$ and range $[0, \infty)$.

To find f^{-1} we write

$$y = \sqrt{x+3} \quad \text{where } x \geq -3, y \geq 0$$

$$x = \sqrt{y+3} \quad \text{where } y \geq -3, x \geq 0 \quad \text{Interchange } x \text{ and } y.$$

$$x^2 = y+3 \quad \text{where } y \geq -3, x \geq 0 \quad \text{Square.}$$

$$y = x^2 - 3 \quad \text{where } y \geq -3, x \geq 0 \quad \text{Solve for } y.$$

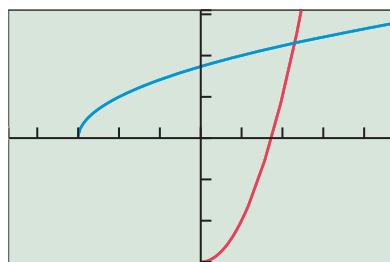
Thus $f^{-1}(x) = x^2 - 3$, with an “inherited” domain restriction of $x \geq 0$. Figure 1.71 shows the two functions. Note the domain restriction of $x \geq 0$ imposed on the parabola $y = x^2 - 3$.

Support Graphically

Use a grapher in parametric mode and compare the graphs of the two sets of parametric equations with Figure 1.71:

$$\begin{array}{lll} x = t & \text{and} & x = \sqrt{t+3} \\ y = \sqrt{t+3} & & y = t \end{array}$$

Now try Exercise 17.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

FIGURE 1.71 The graph of $f(x) = \sqrt{x+3}$ and its inverse, a restricted version of $y = x^2 - 3$. (Example 7)

QUICK REVIEW 1.5 (For help, go to Section P.3 and P.4.)

In Exercises 1–10, solve the equation for y .

1. $x = 3y - 6$

2. $x = 0.5y + 1$

7. $x = \frac{2y + 1}{y - 4}$

8. $x = \frac{4y + 3}{3y - 1}$

3. $x = y^2 + 4$

4. $x = y^2 - 6$

9. $x = \sqrt{y + 3}, y \geq -3$

10. $x = \sqrt{y - 2}, y \geq 2$

5. $x = \frac{y - 2}{y + 3}$

6. $x = \frac{3y - 1}{y + 2}$

SECTION 1.5 EXERCISES

In Exercises 1–4, find the (x, y) pair for the value of the parameter.

1. $x = 3t$ and $y = t^2 + 5$ for $t = 2$

2. $x = 5t - 7$ and $y = 17 - 3t$ for $t = -2$

3. $x = t^3 - 4t$ and $y = \sqrt{t + 1}$ for $t = 3$

4. $x = |t + 3|$ and $y = 1/t$ for $t = -8$

In Exercises 5–8, complete the following. (a) Find the points determined by $t = -3, -2, -1, 0, 1, 2$, and 3. (b) Find a direct algebraic relationship between x and y and determine whether the parametric equations determine y as a function of x . (c) Graph the relationship in the xy -plane.

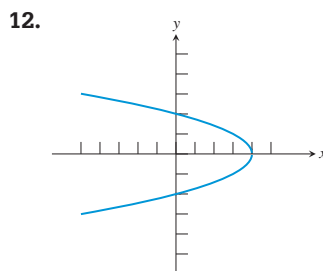
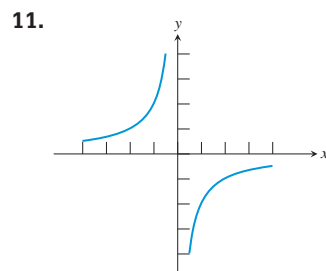
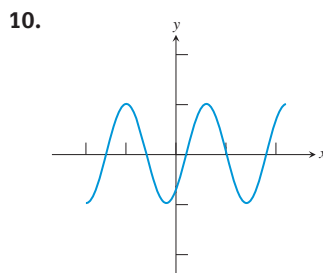
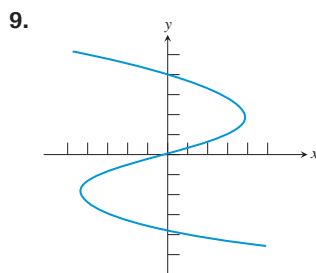
5. $x = 2t$ and $y = 3t - 1$

6. $x = t + 1$ and $y = t^2 - 2t$

7. $x = t^2$ and $y = t - 2$

8. $x = \sqrt{t}$ and $y = 2t - 5$

In Exercises 9–12, the graph of a relation is shown. (a) Is the relation a function? (b) Does the relation have an inverse that is a function?



In Exercises 13–22, find a formula for $f^{-1}(x)$. Give the domain of f^{-1} , including any restrictions “inherited” from f .

13. $f(x) = 3x - 6$

14. $f(x) = 2x + 5$

15. $f(x) = \frac{2x - 3}{x + 1}$

16. $f(x) = \frac{x + 3}{x - 2}$

17. $f(x) = \sqrt{x - 3}$

18. $f(x) = \sqrt{x + 2}$

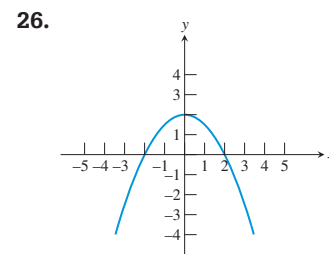
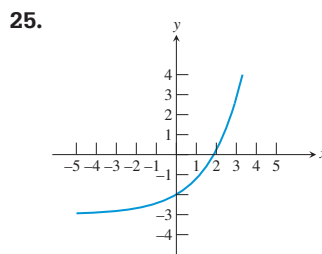
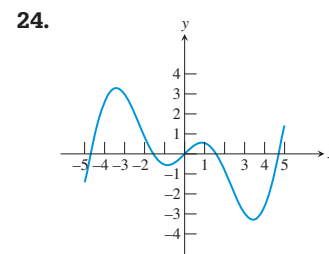
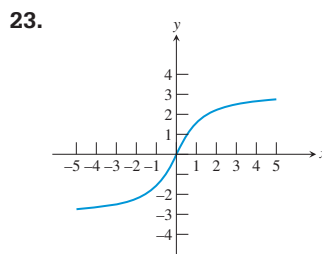
19. $f(x) = x^3$

20. $f(x) = x^3 + 5$

21. $f(x) = \sqrt[3]{x + 5}$

22. $f(x) = \sqrt[3]{x - 2}$

In Exercises 23–26, determine whether the function is one-to-one. If it is one-to-one, sketch the graph of the inverse.



In Exercises 27–32, confirm that f and g are inverses by showing that $f(g(x)) = x$ and $g(f(x)) = x$.

27. $f(x) = 3x - 2$ and $g(x) = \frac{x+2}{3}$

28. $f(x) = \frac{x+3}{4}$ and $g(x) = 4x - 3$

29. $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$

30. $f(x) = \frac{7}{x}$ and $g(x) = \frac{7}{x}$

31. $f(x) = \frac{x+1}{x}$ and $g(x) = \frac{1}{x-1}$

32. $f(x) = \frac{x+3}{x-2}$ and $g(x) = \frac{2x+3}{x-1}$

33. **Currency Conversion** In May of 2002 the exchange rate for converting U.S. dollars (x) to euros (y) was $y = 1.08x$.

(a) How many euros could you get for \$100 U.S.?

(b) What is the inverse function, and what conversion does it represent?

(c) In the spring of 2002, a tourist had an elegant lunch in Provence, France ordering from a “fixed price” 48-euro menu. How much was that in U.S. dollars?

34. **Temperature Conversion** The formula for converting Celsius temperature (x) to Kelvin temperature is $k(x) = x + 273.16$. The formula for converting Fahrenheit temperature (x) to Celsius temperature is $c(x) = (5/9)(x - 32)$.

(a) Find a formula for $c^{-1}(x)$. What is this formula used for?

(b) Find $(k \circ c)(x)$. What is this formula used for?

35. Which pairs of basic functions (Section 1.3) are inverses of each other?

36. Which basic functions (Section 1.3) are their own inverses?

37. Which basic function can be defined parametrically as follows?

$$x = t^3 \text{ and } y = \sqrt{t^6} \text{ for } -\infty < t < \infty$$

38. Which basic function can be defined parametrically as follows?

$$x = 8t^3 \text{ and } y = (2t)^3 \text{ for } -\infty < t < \infty$$

Standardized Test Questions

39. **True or False** If f is a one-to-one function with domain D and range R , then f^{-1} is a one-to-one function with domain R and range D . Justify your answer.

40. **True or False** The set of points $(t + 1, 2t + 3)$ for all real numbers t form a line with slope 2. Justify your answer.

41. **Multiple Choice** Which ordered pair is in the *inverse* of the relation given by $x^2y + 5y = 9$?

- (A) (2, 1) (B) (-2, 1) (C) (-1, 2) (D) (2, -1)
(E) (1, -2)

42. **Multiple Choice** Which ordered pair is not in the *inverse* of the relation given by $xy^2 - 3x = 12$?

- (A) (0, -4) (B) (4, 1) (C) (3, 2) (D) (2, 12)
(E) (1, -6)

43. **Multiple Choice** Which function is the *inverse* of the function $f(x) = 3x - 2$?

- (A) $g(x) = \frac{x}{3} + 2$
(B) $g(x) = 2 - 3x$
(C) $g(x) = \frac{x+2}{3}$
(D) $g(x) = \frac{x-3}{2}$
(E) $g(x) = \frac{x-2}{3}$

44. **Multiple Choice** Which function is the *inverse* of the function $f(x) = x^3 + 1$?

- (A) $g(x) = \sqrt[3]{x-1}$
(B) $g(x) = \sqrt[3]{x} - 1$
(C) $g(x) = x^3 - 1$
(D) $g(x) = \sqrt[3]{x+1}$
(E) $g(x) = 1 - x^3$

Explorations

45. **Function Properties Inherited by Inverses** There are some properties of functions that are automatically shared by inverse functions (when they exist) and some that are not. Suppose that f has an inverse function f^{-1} . Give an algebraic or graphical argument (not a rigorous formal proof) to show that each of these properties of f must necessarily be shared by f^{-1} .

- (a) f is continuous.
(b) f is one-to-one.
(c) f is odd (graphically, symmetric with respect to the origin).
(d) f is increasing.

46. **Function Properties Not Inherited by Inverses** There are some properties of functions that are not necessarily shared by inverse functions, even if the inverses exist. Suppose that f has an inverse function f^{-1} . For each of the following properties, give an example to show that f can have the property while f^{-1} does not.

- (a) f has a graph with a horizontal asymptote.
(b) f has domain all real numbers.
(c) f has a graph that is bounded above.
(d) f has a removable discontinuity at $x = 5$.

- 47. Scaling Algebra Grades** A teacher gives a challenging algebra test to her class. The lowest score is 52, which she decides to scale to 70. The highest score is 88, which she decides to scale to 97.
- (a) Using the points (52, 70) and (88, 97), find a linear equation that can be used to convert raw scores to scaled grades.
- (b) Find the inverse of the function defined by this linear equation. What does the inverse function do?
- 48. Writing to Learn** (Continuation of Exercise 47) Explain why it is important for fairness that the scaling function used by the teacher be an *increasing* function. (Caution: It is *not* because “everyone’s grade must go up.” What would the scaling function in Exercise 47 do for a student who does enough “extra credit” problems to get a raw score of 136?)

Extending the Ideas

- 49. Modeling a Fly Ball Parametrically** A baseball that leaves the bat at an angle of 60° from horizontal traveling 110 feet per second follows a path that can be modeled by the following pair of parametric equations. (You might enjoy verifying this if you have studied motion in physics.)

$$x = 110(t)\cos(60^\circ)$$

$$y = 110(t)\sin(60^\circ) - 16t^2$$

You can simulate the flight of the ball on a grapher. Set your grapher to parametric mode and put the functions above in for X2T and Y2T. Set X1T = 325 and Y1T = 5T to draw a 30-foot fence 325 feet from home plate. Set Tmin = 0, Tmax = 6, Tstep = 0.1, Xmin = 0, Xmax = 350, Xscl = 0, Ymin = 0, Ymax = 300, and Yscl = 0.

- (a) Now graph the function. Does the fly ball clear the fence?
- (b) Change the angle to 30° and run the simulation again. Does the ball clear the fence?
- (c) What angle is optimal for hitting the ball? Does it clear the fence when hit at that angle?

- 50. The Baylor GPA Scale Revisited** (See Problem 78 in Section 1.2.) The function used to convert Baylor School percentage grades to GPAs on a 4-point scale is

$$y = \left(\frac{3^{1.7}}{30} (x - 65) \right)^{\frac{1}{1.7}} + 1.$$

The function has domain $[65, 100]$. Anything below 65 is a failure and automatically converts to a GPA of 0.

- (a) Find the inverse function algebraically. What can the inverse function be used for?
- (b) Does the inverse function have any domain restrictions?
- (c) Verify with a graphing calculator that the function found in (a) and the given function are really inverses.
- 51. Group Activity** (Continuation of Exercise 50) The number 1.7 that appears in two places in the GPA scaling formula is called the scaling factor (k). The value of k can be changed to alter the curvature of the graph while keeping the points (65, 1) and (95, 4) fixed. It was felt that the lowest D (65) needed to be scaled to 1.0, while the middle A (95) needed to be scaled to 4.0. The faculty’s Academic Council considered several values of k before settling on 1.7 as the number that gives the “fairest” GPAs for the other percentage grades.
- Try changing k to other values between 1 and 2. What kind of scaling curve do you get when $k = 1$? Do you agree with the Baylor decision that $k = 1.7$ gives the fairest GPAs?

1.6

Graphical Transformations

What you'll learn about

- Transformations
- Vertical and Horizontal Translations
- Reflections Across Axes
- Vertical and Horizontal Stretches and Shrinks
- Combining Transformations

... and why

Studying transformations will help you to understand the relationships between graphs that have similarities but are not the same.

Transformations

The following functions are all different:

$$y = x^2$$

$$y = (x - 3)^2$$

$$y = 1 - x^2$$

$$y = x^2 - 4x + 5$$

However, a look at their graphs shows that, while no two are exactly the same, all four have the same identical *shape* and *size*. Understanding how algebraic alterations change the shapes, sizes, positions, and orientations of graphs is helpful for understanding the connection between algebraic and graphical models of functions.

In this section we relate graphs using **transformations**, which are functions that map real numbers to real numbers. By acting on the x -coordinates and y -coordinates of points, transformations change graphs in predictable ways. **Rigid transformations**, which leave the size and shape of a graph unchanged, include horizontal translations, vertical translations, reflections, or any combination of these. **Non-rigid transformations**, which generally distort the shape of a graph, include horizontal or vertical stretches and shrinks.

Vertical and Horizontal Translations

A **vertical translation** of the graph of $y = f(x)$ is a shift of the graph up or down in the coordinate plane. A **horizontal translation** is a shift of the graph to the left or the right. The following exploration will give you a good feel for what translations are and how they occur.

EXPLORATION 1 Introducing Translations

Set your viewing window to $[-5, 5]$ by $[-5, 15]$ and your graphing mode to sequential as opposed to simultaneous.

1. Graph the functions

$$y_1 = x^2$$

$$y_4 = y_1(x) - 2 = x^2 - 2$$

$$y_2 = y_1(x) + 3 = x^2 + 3$$

$$y_5 = y_1(x) - 4 = x^2 - 4$$

$$y_3 = y_1(x) + 1 = x^2 + 1$$

on the same screen. What effect do the $+3$, $+1$, -2 , and -4 seem to have?

2. Graph the functions

$$y_1 = x^2$$

$$y_4 = y_1(x - 2) = (x - 2)^2$$

$$y_2 = y_1(x + 3) = (x + 3)^2$$

$$y_5 = y_1(x - 4) = (x - 4)^2$$

$$y_3 = y_1(x + 1) = (x + 1)^2$$

on the same screen. What effect do the $+3$, $+1$, -2 , and -4 seem to have?

3. Repeat steps 1 and 2 for the functions $y_1 = x^3$, $y_1 = |x|$, and $y_1 = \sqrt{x}$. Do your observations agree with those you made after steps 1 and 2?

TECHNOLOGY ALERT

In Exploration 1, the notation $y_1(x + 3)$ means the function y_1 , evaluated at $x + 3$. It does not mean multiplication.

In general, replacing x by $x - c$ shifts the graph horizontally c units. Similarly, replacing y by $y - c$ shifts the graph vertically c units. If c is positive the shift is to the right or up; if c is negative the shift is to the left or down.

This is a nice, consistent rule that unfortunately gets complicated by the fact that the c for a vertical shift rarely shows up being subtracted from y . Instead, it usually shows up on the other side of the equal sign being *added* to $f(x)$. That leads us to the following rule, which only *appears* to be different for horizontal and vertical shifts:

Translations

Let c be a positive real number. Then the following transformations result in translations of the graph of $y = f(x)$:

Horizontal Translations

$y = f(x - c)$ a translation to the right by c units

$y = f(x + c)$ a translation to the left by c units

Vertical Translations

$y = f(x) + c$ a translation up by c units

$y = f(x) - c$ a translation down by c units

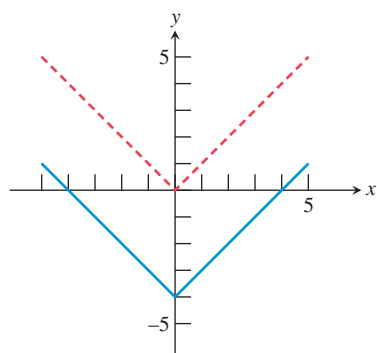


FIGURE 1.72 $y = |x| - 4$ (Example 1)

EXAMPLE 1 Vertical Translations

Describe how the graph of $y = |x|$ can be transformed to the graph of the given equation.

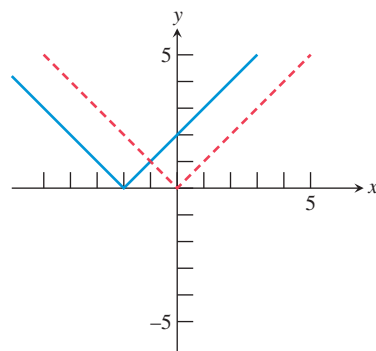
(a) $y = |x| - 4$ (b) $y = |x + 2|$

SOLUTION

(a) The equation is in the form $y = f(x) - 4$, a translation down by 4 units. See Figure 1.72.

(b) The equation is in the form $y = f(x + 2)$, a translation left by 2 units. See Figure 1.73 on the next page.

Now try Exercise 3.

FIGURE 1.73 $y = |x + 2|$ (Example 1)**EXAMPLE 2 Finding Equations for Translations**

Each view in Figure 1.74 shows the graph of $y_1 = x^3$ and a vertical or horizontal translation y_2 . Write an equation for y_2 as shown in each graph.

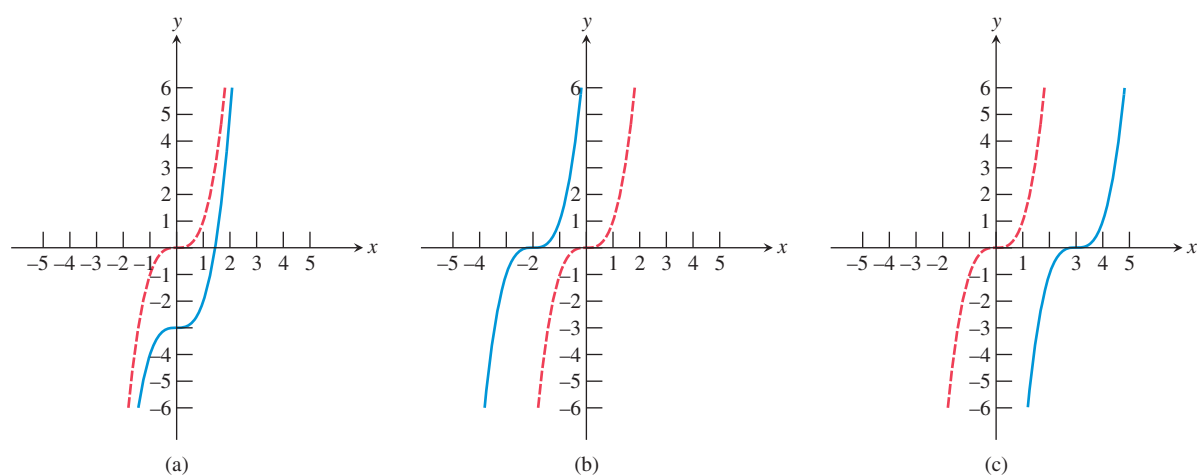
SOLUTION

(a) $y_2 = x^3 - 3 = y_1(x) - 3$ (a vertical translation down by 3 units)

(b) $y_2 = (x + 2)^3 = y_1(x + 2)$ (a horizontal translation left by 2 units)

(c) $y_2 = (x - 3)^3 = y_1(x - 3)$ (a horizontal translation right by 3 units)

Now try Exercise 25.

FIGURE 1.74 Translations of $y_1 = x^3$. (Example 2)

Reflections Across Axes

Points (x, y) and $(x, -y)$ are **reflections of each other across the x -axis**. Points (x, y) and $(-x, y)$ are **reflections of each other across the y -axis**. (See Figure 1.75.) Two points (or graphs) that are symmetric with respect to a line are **reflections of each other across that line**.

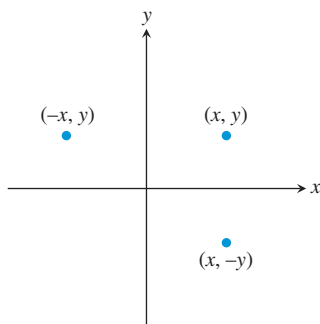


FIGURE 1.75 The point (x, y) and its reflections across the x - and y -axes.

Figure 1.75 suggests that a reflection across the x -axis results when y is replaced by $-y$, and a reflection across the y -axis results when x is replaced by $-x$.

Reflections

The following transformations result in reflections of the graph of $y = f(x)$:

Across the x -axis

$$y = -f(x)$$

Across the y -axis

$$y = f(-x)$$

EXAMPLE 3 Finding Equations for Reflections

Find an equation for the reflection of $f(x) = \frac{5x - 9}{x^2 + 3}$ across each axis.

SOLUTION

Solve Algebraically

$$\text{Across the } x\text{-axis: } y = -f(x) = -\frac{5x - 9}{x^2 + 3} = \frac{9 - 5x}{x^2 + 3}$$

$$\text{Across the } y\text{-axis: } y = f(-x) = \frac{5(-x) - 9}{(-x)^2 + 3} = \frac{-5x - 9}{x^2 + 3}$$

Support Graphically

The graphs in Figure 1.76 support our algebraic work.

continued

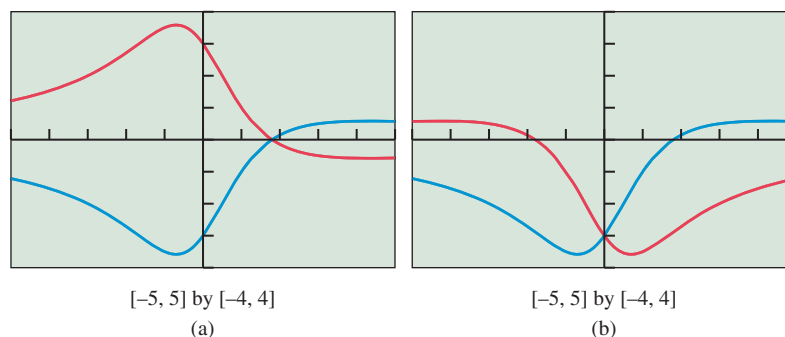


FIGURE 1.76 Reflections of $f(x) = (5x - 9)/(x^2 + 3)$ across (a) the x -axis and (b) the y -axis. (Example 3)

Now try Exercise 29.

You might expect that odd and even functions, whose graphs already possess special symmetries, would exhibit special behavior when reflected across the axes. They do, as shown by Example 4 and Exercises 33 and 34.

EXAMPLE 4 Reflecting Even Functions

Prove that the graph of an even function remains unchanged when it is reflected across the y -axis.

SOLUTION

Note that we can get plenty of graphical support for these statements by reflecting the graphs of various even functions, but what is called for here is **proof**, which will require algebra.

Let f be an even function; that is, $f(-x) = f(x)$ for all x in the domain of f . To reflect the graph of $y = f(x)$ across the y -axis, we make the transformation $y = f(-x)$. But $f(-x) = f(x)$ for all x in the domain of f , so this transformation results in $y = f(x)$. The graph of f therefore remains unchanged.

Now try Exercise 33.

Graphing Absolute Value Compositions

Given the graph of $y = f(x)$,

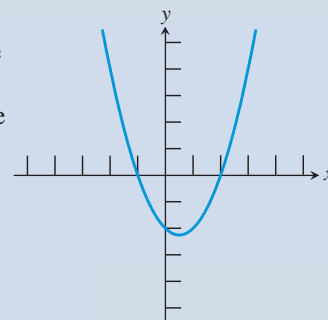
the graph of $y = |f(x)|$ can be obtained by reflecting the portion of the graph below the x -axis across the x -axis, leaving the portion above the x -axis unchanged;

the graph of $y = f(|x|)$ can be obtained by *replacing* the portion of the graph to the left of the y -axis by a reflection of the portion to the right of the y -axis across the y -axis, leaving the portion to the right of the y -axis unchanged. (The result will show even symmetry.)

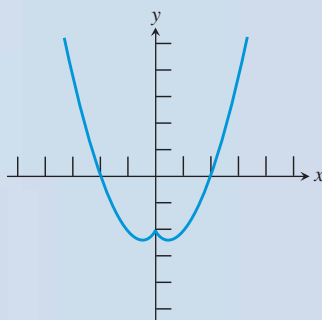
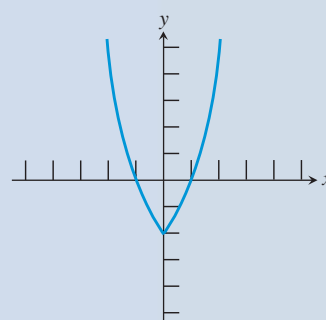
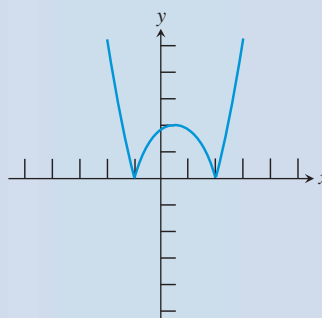
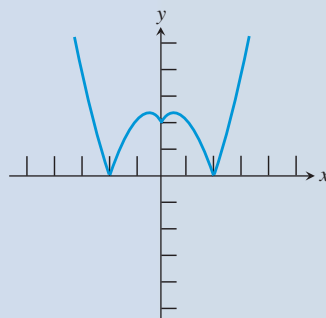
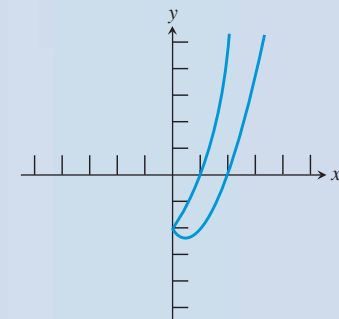
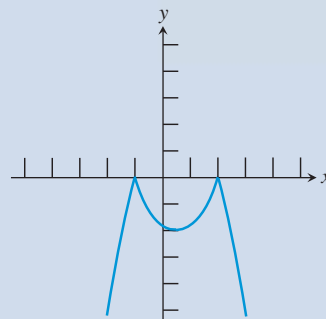
Function compositions with absolute value can be realized graphically by reflecting portions of graphs, as you will see in the following Exploration.

EXPLORATION 2 Compositions with Absolute Value

The graph of $y = f(x)$ is shown at the right. Match each of the graphs below with one of the following equations and use the language of function reflection to defend your match. Note that two of the graphs will not be used.



1. $y = |f(x)|$
2. $y = f(|x|)$
3. $y = -|f(x)|$
4. $y = |f(|x|)|$

(A)**(B)****(C)****(D)****(E)****(F)**

Vertical and Horizontal Stretches and Shrinks

We now investigate what happens when we multiply all the y -coordinates (or all the x -coordinates) of a graph by a fixed real number.

EXPLORATION 3 Introducing Stretches and Shrinks

Set your viewing window to $[-4.7, 4.7]$ by $[-1.1, 5.1]$ and your graphing mode to sequential as opposed to simultaneous.

1. Graph the functions

$$y_1 = \sqrt{4 - x^2}$$

$$y_2 = 1.5y_1(x) = 1.5\sqrt{4 - x^2}$$

$$y_3 = 2y_1(x) = 2\sqrt{4 - x^2}$$

$$y_4 = 0.5y_1(x) = 0.5\sqrt{4 - x^2}$$

$$y_5 = 0.25y_1(x) = 0.25\sqrt{4 - x^2}$$

on the same screen. What effect do the 1.5, 2, 0.5, and 0.25 seem to have?

2. Graph the functions

$$y_1 = \sqrt{4 - x^2}$$

$$y_2 = y_1(1.5x) = \sqrt{4 - (1.5x)^2}$$

$$y_3 = y_1(2x) = \sqrt{4 - (2x)^2}$$

$$y_4 = y_1(0.5x) = \sqrt{4 - (0.5x)^2}$$

$$y_5 = y_1(0.25x) = \sqrt{4 - (0.25x)^2}$$

on the same screen. What effect do the 1.5, 2, 0.5, and 0.25 seem to have?

Exploration 3 suggests that multiplication of x or y by a constant results in a horizontal or vertical stretching or shrinking of the graph.

In general, *replacing x by x/c* distorts the graph horizontally by a factor of c . Similarly, *replacing y by y/c* distorts the graph vertically by a factor of c . If c is greater than 1 the distortion is a stretch; if c is less than 1 the distortion is a shrink.

As with translations, this is a nice, consistent rule that unfortunately gets complicated by the fact that the c for a vertical stretch or shrink rarely shows up as a divisor of y . Instead, it usually shows up on the other side of the equal sign as a *factor* multiplied by $f(x)$. That leads us to the following rule:

Stretches and Shrinks

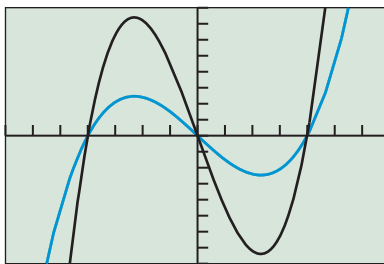
Let c be a positive real number. Then the following transformations result in stretches or shrinks of the graph of $y = f(x)$:

Horizontal Stretches or Shrinks

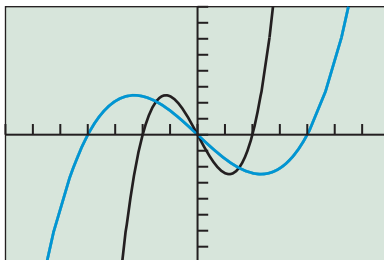
$$y = f\left(\frac{x}{c}\right) \quad \begin{cases} \text{a stretch by a factor of } c & \text{if } c > 1 \\ \text{a shrink by a factor of } c & \text{if } c < 1 \end{cases}$$

Vertical Stretches or Shrinks

$$y = c \cdot f(x) \quad \begin{cases} \text{a stretch by a factor of } c & \text{if } c > 1 \\ \text{a shrink by a factor of } c & \text{if } c < 1 \end{cases}$$



[-7, 7] by [-80, 80]
(a)



[-7, 7] by [-80, 80]
(b)

FIGURE 1.77 The graph of $y_1 = f(x) = x^3 - 16x$, shown with (a) a vertical stretch and (b) a horizontal shrink. (Example 5)

EXAMPLE 5 Finding Equations for Stretches and Shrinks

Let C_1 be the curve defined by $y_1 = f(x) = x^3 - 16x$. Find equations for the following non-rigid transformations of C_1 :

- (a) C_2 is a vertical stretch of C_1 by a factor of 3.
- (b) C_3 is a horizontal shrink of C_1 by a factor of $1/2$.

SOLUTION

Solve Algebraically

- (a) Denote the equation for C_2 by y_2 . Then

$$\begin{aligned} y_2 &= 3 \cdot f(x) \\ &= 3(x^3 - 16x) \\ &= 3x^3 - 48x \end{aligned}$$

- (b) Denote the equation for C_3 by y_3 . Then

$$\begin{aligned} y_3 &= f\left(\frac{x}{1/2}\right) \\ &= f(2x) \\ &= (2x)^3 - 16(2x) \\ &= 8x^3 - 32x \end{aligned}$$

Support Graphically

The graphs in Figure 1.77 support our algebraic work.

Now try Exercise 39.

Combining Transformations

Transformations may be performed in succession—one after another. If the transformations include stretches, shrinks, or reflections, the order in which the transformations are performed may make a difference. In those cases, be sure to pay particular attention to order.

EXAMPLE 6 Combining Transformations in Order

(a) The graph of $y = x^2$ undergoes the following transformations, in order. Find the equation of the graph that results.

- a horizontal shift 2 units to the right
- a vertical stretch by a factor of 3
- a vertical translation 5 units up

(b) Apply the transformations in (a) in the opposite order and find the equation of the graph that results.

SOLUTION

(a) Applying the transformations in order, we have

$$x^2 \Rightarrow (x - 2)^2 \Rightarrow 3(x - 2)^2 \Rightarrow 3(x - 2)^2 + 5$$

Expanding the final expression, we get the function $y = 3x^2 - 12x + 17$.

(b) Applying the transformations in the opposite order, we have

$$x^2 \Rightarrow x^2 + 5 \Rightarrow 3(x^2 + 5) \Rightarrow 3((x - 2)^2 + 5)$$

Expanding the final expression, we get the function $y = 3x^2 - 12x + 27$.

The second graph is ten units higher than the first graph because the vertical stretch lengthens the vertical translation when the translation occurs first. Order often matters when stretches, shrinks, or reflections are involved.

Now try Exercise 47.

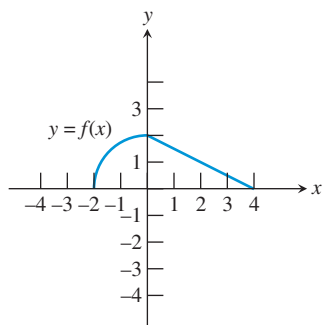


FIGURE 1.78 The graph of the function $y = f(x)$ in Example 7.

**EXAMPLE 7** Transforming a Graph Geometrically

The graph of $y = f(x)$ is shown in Figure 1.78. Determine the graph of the composite function $y = 2f(x + 1) - 3$ by showing the effect of a sequence of transformations on the graph of $y = f(x)$.

SOLUTION

The graph of $y = 2f(x + 1) - 3$ can be obtained from the graph of $y = f(x)$ by the following sequence of transformations:

- a vertical stretch by a factor of 2 to get $y = 2f(x)$ (Figure 1.79a)
- a horizontal translation 1 unit to the left to get $y = 2f(x + 1)$ (Figure 1.79b)
- a vertical translation 3 units down to get $y = 2f(x + 1) - 3$ (Figure 1.79c)

(The order of the first two transformations can be reversed without changing the final graph.)

Now try Exercise 51.

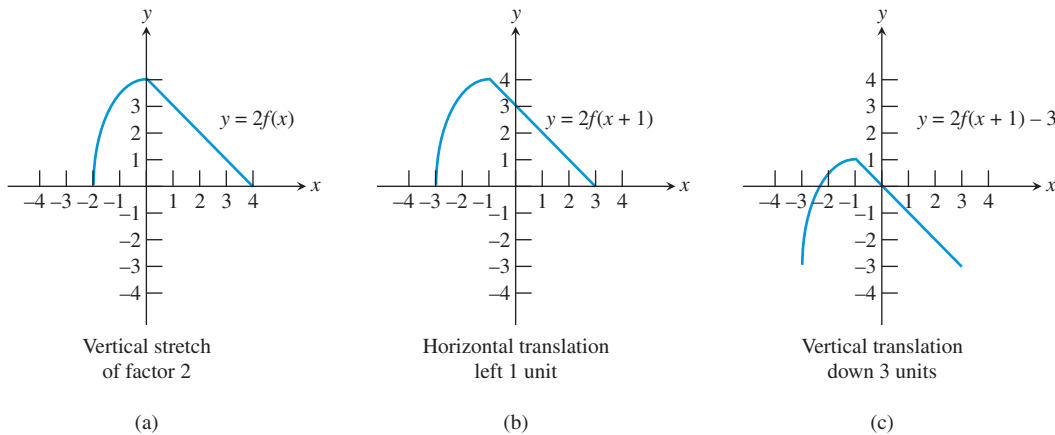


FIGURE 1.79 Transforming the graph of $y = f(x)$ in Figure 1.78 to get the graph of $y = 2f(x + 1) - 3$. (Example 7)

QUICK REVIEW 1.6 (For help, go to Section A.2.)

In Exercises 1–6, write the expression as a binomial squared.

1. $x^2 + 2x + 1$
2. $x^2 - 6x + 9$
3. $x^2 + 12x + 36$
4. $4x^2 + 4x + 1$
5. $x^2 - 5x + \frac{25}{4}$
6. $4x^2 - 20x + 25$

In Exercises 7–10, perform the indicated operations and simplify.

7. $(x - 2)^2 + 3(x - 2) + 4$
8. $2(x + 3)^2 - 5(x + 3) - 2$
9. $(x - 1)^3 + 3(x - 1)^2 - 3(x - 1)$
10. $2(x + 1)^3 - 6(x + 1)^2 + 6(x + 1) - 2$

SECTION 1.6 EXERCISES

In Exercises 1–8, describe how the graph of $y = x^2$ can be transformed to the graph of the given equation.

1. $y = x^2 - 3$
2. $y = x^2 + 5.2$
3. $y = (x + 4)^2$
4. $y = (x - 3)^2$
5. $y = (100 - x)^2$
6. $y = x^2 - 100$
7. $y = (x - 1)^2 + 3$
8. $y = (x + 50)^2 - 279$

In Exercises 9–12, describe how the graph of $y = \sqrt{x}$ can be transformed to the graph of the given equation.

9. $y = -\sqrt{x}$
10. $y = \sqrt{x - 5}$
11. $y = \sqrt{-x}$
12. $y = \sqrt{3 - x}$

In Exercises 13–16, describe how the graph of $y = x^3$ can be transformed to the graph of the given equation.

13. $y = 2x^3$
14. $y = (2x)^3$
15. $y = (0.2x)^3$
16. $y = 0.3x^3$

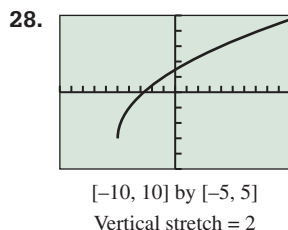
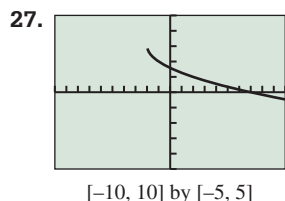
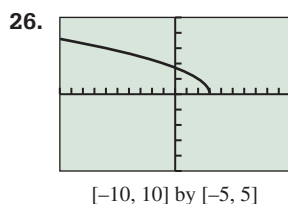
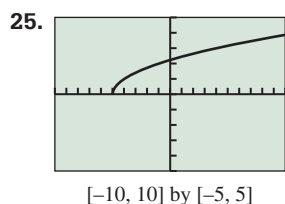
In Exercises 17–20, describe how to transform the graph of f into the graph of g .

17. $f(x) = \sqrt{x + 2}$ and $g(x) = \sqrt{x - 4}$
18. $f(x) = (x - 1)^2$ and $g(x) = -(x + 3)^2$
19. $f(x) = (x - 2)^3$ and $g(x) = -(x + 2)^3$
20. $f(x) = |2x|$ and $g(x) = 4|x|$

In Exercises 21–24, sketch the graphs of f , g , and h by hand. Support your answers with a grapher.

21. $f(x) = (x + 2)^2$
 $g(x) = 3x^2 - 2$
 $h(x) = -2(x - 3)^2$
22. $f(x) = x^3 - 2$
 $g(x) = (x + 4)^3 - 1$
 $h(x) = 2(x - 1)^3$
23. $f(x) = \sqrt[3]{x + 1}$
 $g(x) = 2\sqrt[3]{x} - 2$
 $h(x) = -\sqrt[3]{x - 3}$
24. $f(x) = -2|x| - 3$
 $g(x) = 3|x + 5| + 4$
 $h(x) = |3x|$

In Exercises 25–28, the graph is that of a function $y = f(x)$ that can be obtained by transforming the graph of $y = \sqrt{x}$. Write a formula for the function f .



In Exercises 29–32, find the equation of the reflection of f across (a) the x -axis and (b) the y -axis.

29. $f(x) = x^3 - 5x^2 - 3x + 2$ 30. $f(x) = 2\sqrt{x+3} - 4$

31. $f(x) = \sqrt[3]{8x}$ 32. $f(x) = 3|x+5|$

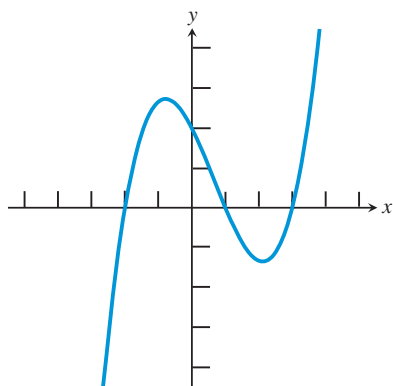
33. Reflecting Odd Functions Prove that the graph of an odd function is the same when reflected across the x -axis as it is when reflected across the y -axis.

34. Reflecting Odd Functions Prove that if an odd function is reflected about the y -axis and then reflected again about the x -axis, the result is the original function.

Exercises 35–38 refer to the graph of $y = f(x)$ shown below. In each case, sketch a graph of the new function.

35. $y = |f(x)|$ 36. $y = f(|x|)$

37. $y = -f(|x|)$ 38. $y = |f(|x|)|$



In Exercises 39–42, transform the given function by (a) a vertical stretch by a factor of 2, and (b) a horizontal shrink by a factor of $1/3$.

39. $f(x) = x^3 - 4x$

40. $f(x) = |x+2|$

41. $f(x) = x^2 + x - 2$

42. $f(x) = \frac{1}{x+2}$

In Exercises 43–46, describe a basic graph and a sequence of transformations that can be used to produce a graph of the given function.

43. $y = 2(x-3)^2 - 4$

44. $y = -3\sqrt{x+1}$

45. $y = (3x)^2 - 4$

46. $y = -2|x+4| + 1$

In Exercises 47–50, a graph G is obtained from a graph of y by the sequence of transformations indicated. Write an equation whose graph is G .

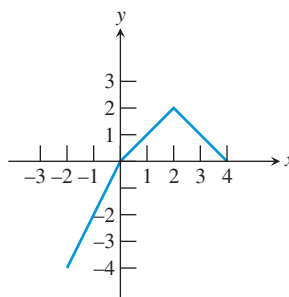
47. $y = x^2$: a vertical stretch by a factor of 3, then a shift right 4 units.

48. $y = x^2$: a shift right 4 units, then a vertical stretch by a factor of 3.

49. $y = |x|$: a shift left 2 units, then a vertical stretch by a factor of 2, and finally a shift down 4 units.

50. $y = |x|$: a shift left 2 units, then a horizontal shrink by a factor of $1/2$, and finally a shift down 4 units.

Exercises 51–54 refer to the function f whose graph is shown below.



51. Sketch the graph of $y = 2 + 3f(x+1)$.

52. Sketch the graph of $y = -f(x+1) + 1$.

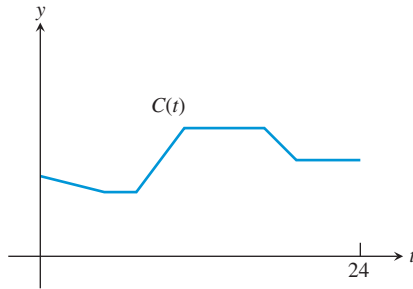
53. Sketch the graph of $y = f(2x)$.

54. Sketch the graph of $y = 2f(x-1) + 2$.

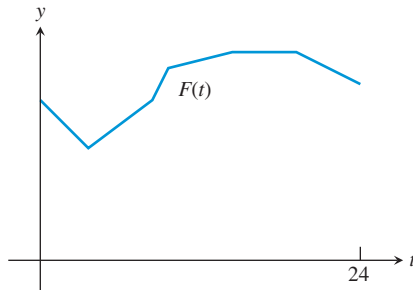
55. Writing to Learn Graph some examples to convince yourself that a reflection and a translation can have a different effect when combined in one order than when combined in the opposite order. Then explain in your own words why this can happen.

56. Writing to Learn Graph some examples to convince yourself that vertical stretches and shrinks do not affect a graph's x -intercepts. Then explain in your own words why this is so.

- 57. Celsius vs. Fahrenheit** The graph shows the temperature in degrees Celsius in Windsor, Ontario, for one 24-hour period. Describe the transformations that convert this graph to one showing degrees Fahrenheit. [Hint: $F(t) = (9/5)C(t) + 32$.]



- 58. Fahrenheit vs. Celsius** The graph shows the temperature in degrees Fahrenheit in Mt. Clemens, Michigan, for one 24-hour period. Describe the transformations that convert this graph to one showing degrees Celsius. [Hint: $F(t) = (9/5)C(t) + 32$.]



Standardized Test Questions

- 59. True or False** The function $y = f(x + 3)$ represents a translation to the right by 3 units of the graph of $y = f(x)$. Justify your answer.
- 60. True or False** The function $y = f(x) - 4$ represents a translation down 4 units of the graph of $y = f(x)$. Justify your answer.
- In Exercises 61–64, you may use a graphing calculator to answer the question.
- 61. Multiple Choice** Given a function f , which of the following represents a vertical stretch by a factor of 3?
- (A) $y = f(3x)$ (B) $y = f(x/3)$
 (C) $y = 3f(x)$ (D) $y = f(x)/3$
 (E) $y = f(x) + 3$
- 62. Multiple Choice** Given a function f , which of the following represents a horizontal translation of 4 units to the right?
- (A) $y = f(x) + 4$ (B) $y = f(x) - 4$
 (C) $y = f(x + 4)$ (D) $y = f(x - 4)$
 (E) $y = 4f(x)$

- 63. Multiple Choice** Given a function f , which of the following represents a vertical translation of 2 units upward, followed by a reflection across the y -axis?

- (A) $y = f(-x) + 2$ (B) $y = 2 - f(x)$
 (C) $y = f(2 - x)$ (D) $y = -f(x - 2)$
 (E) $y = f(x) - 2$

- 64. Multiple Choice** Given a function f , which of the following represents reflection across the x -axis, followed by a horizontal shrink by a factor of $1/2$?

- (A) $y = -2f(x)$ (B) $y = -f(x)/2$
 (C) $y = f(-2x)$ (D) $y = -f(x/2)$
 (E) $y = -f(2x)$

Explorations

- 65. International Finance** Table 1.11 shows the price of a share of stock in Dell Computer for the first eight months of 2004:



Table 1.11 Dell Computer

Month	Price (\$)
1	33.44
2	32.65
3	33.62
4	34.78
5	35.24
6	35.82
7	35.47
8	34.84

Source: Yahoo! Finance

- (a) Graph price (y) as a function of month (x) as a line graph, connecting the points to make a continuous graph.
- (b) Explain what transformation you would apply to this graph to produce a graph showing the price of the stock in Japanese yen.

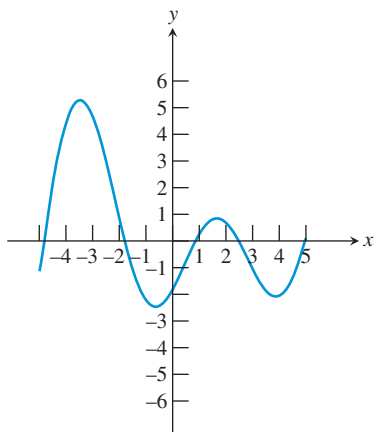


- 66. Group Activity** Get with a friend and graph the function $y = x^2$ on both your graphers. Apply a horizontal or vertical stretch or shrink to the function on one of the graphers. Then change the window of that grapher to make the two graphs look the same. Can you formulate a general rule for how to find the window?

Extending the Ideas

67. The Absolute Value Transformation Graph the function $f(x) = x^4 - 5x^3 + 4x^2 + 3x + 2$ in the viewing window $[-5, 5]$ by $[-10, 10]$. (Put the equation in Y1.)

- (a) Study the graph and try to predict what the graph of $y = |f(x)|$ will look like. Then turn Y1 off and graph $Y2 = \text{abs}(Y1)$. Did you predict correctly?
- (b) Study the original graph again and try to predict what the graph of $y = f(|x|)$ will look like. Then turn Y1 off and graph $Y2 = Y1(\text{abs}(X))$. Did you predict correctly?
- (c) Given the graph of $y = g(x)$ shown below, sketch a graph of $y = |g(x)|$.
- (d) Given the graph of $y = g(x)$ shown below, sketch a graph of $y = g(|x|)$.



68. Parametric Circles and Ellipses Set your grapher to parametric and radian mode and your window as follows:

$$T_{\min} = 0, T_{\max} = 7, T_{\text{step}} = 0.1$$

$$X_{\min} = -4.7, X_{\max} = 4.7, X_{\text{scl}} = 1$$

$$Y_{\min} = -3.1, Y_{\max} = 3.1, Y_{\text{scl}} = 1$$

- (a) Graph the parametric equations $x = \cos t$ and $y = \sin t$. You should get a circle of radius 1.
- (b) Use a transformation of the parametric function of x to produce the graph of an ellipse that is 4 units wide and 2 units tall.
- (c) Use a transformation of both parametric functions to produce a circle of radius 3.
- (d) Use a transformation of both functions to produce an ellipse that is 8 units wide and 4 units tall.

(You will learn more about ellipses in Chapter 8.)

1.7

Modeling with Functions

What you'll learn about

- Functions from Formulas
- Functions from Graphs
- Functions from Verbal Descriptions
- Functions from Data

... and why

Using a function to model a variable under observation in terms of another variable often allows one to make predictions in practical situations, such as predicting the future growth of a business based on known data.

Functions from Formulas

Now that you have learned more about what functions are and how they behave, we want to return to the modeling theme of Section 1.1. In that section we stressed that one of the goals of this course was to become adept at using numerical, algebraic, and graphical models of the real world in order to solve problems. We now want to focus your attention more precisely on modeling with *functions*.

You have already seen quite a few formulas in the course of your education. Formulas involving two variable quantities always relate those variables implicitly, and quite often the formulas can be solved to give one variable explicitly as a function of the other. In this book we will use a variety of formulas to pose and solve problems algebraically, although we will not assume prior familiarity with those formulas that we borrow from other subject areas (like physics or economics). We *will* assume familiarity with certain key formulas from mathematics.

EXAMPLE 1 Forming Functions from Formulas

Write the area A of a circle as a function of its

- (a) radius r .
- (b) diameter d .
- (c) circumference C .

SOLUTION

- (a) The familiar area formula from geometry gives A as a function of r :

$$A = \pi r^2.$$

- (b) This formula is not so familiar. However, we know that $r = d/2$, so we can substitute that expression for r in the area formula:

$$A = \pi r^2 = \pi(d/2)^2 = (\pi/4)d^2.$$

- (c) Since $C = 2\pi r$, we can solve for r to get $r = C/(2\pi)$. Then substitute to get A :
 $A = \pi r^2 = \pi(C/(2\pi))^2 = \pi C^2/(4\pi^2) = C^2/(4\pi).$

Now try Exercise 19.

**EXAMPLE 2 A Maximum Value Problem**

A square of side x inches is cut out of each corner of an 8 in. by 15 in. piece of cardboard and the sides are folded up to form an open-topped box (Figure 1.80).

- Write the volume V of the box as a function of x .
- Find the domain of V as a function of x . (Note that the model imposes restrictions on x .)
- Graph V as a function of x over the domain found in part (b) and use the maximum finder on your grapher to determine the maximum volume such a box can hold.
- How big should the cut-out squares be in order to produce the box of maximum volume?

SOLUTION

- The box will have a base with sides of width $8 - 2x$ and length $15 - 2x$. The depth of the box will be x when the sides are folded up. Therefore $V = x(8 - 2x)(15 - 2x)$.
- The formula for V is a polynomial with domain all reals. However, the depth x must be nonnegative, as must the width of the base, $8 - 2x$. Together, these two restrictions yield a domain of $[0, 4]$. (The endpoints give a box with no volume, which is as mathematically feasible as other zero concepts.)

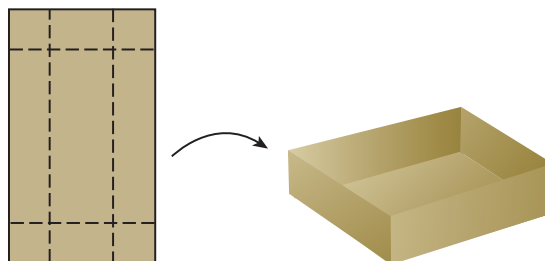
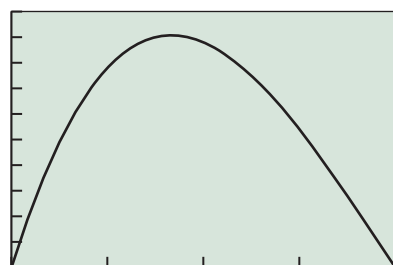


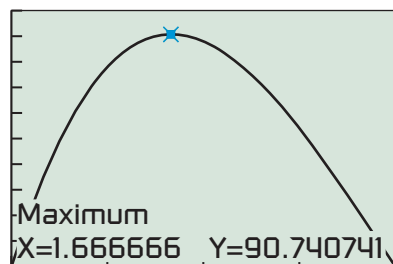
FIGURE 1.80 An open-topped box made by cutting the corners from a piece of cardboard and folding up the sides. (Example 2)

- The graph is shown in Figure 1.81. The maximum finder shows that the maximum occurs at the point $(5/3, 90.74)$. The maximum volume is about 90.74 in.^3 .
- Each square should have sides of one-and-two-thirds inches.

Now try Exercise 33.



$[0, 4]$ by $[0, 100]$
(a)



$[0, 4]$ by $[0, 100]$
(b)

FIGURE 1.81 The graph of the volume of the box in Example 2.

Functions from Graphs

When “thinking graphically” becomes a genuine part of your problem-solving strategy, it is sometimes actually easier to start with the graphical model than it is to go straight to the algebraic formula. The graph provides valuable information about the function.

EXAMPLE 3 Protecting an Antenna

A small satellite dish is packaged with a cardboard cylinder for protection. The parabolic dish is 24 in. in diameter and 6 in. deep, and the diameter of the cardboard cylinder is 12 in. How tall must the cylinder be to fit in the middle of the dish and be flush with the top of the dish? (See Figure 1.82.)

SOLUTION**Solve Algebraically**

The diagram in Figure 1.82a showing the cross section of this 3-dimensional problem is also a 2-dimensional graph of a quadratic function. We can transform our basic function $y = x^2$ with a vertical shrink so that it goes through the points $(12, 6)$ and $(-12, 6)$, thereby producing a graph of the parabola in the coordinate plane (Figure 1.82b).

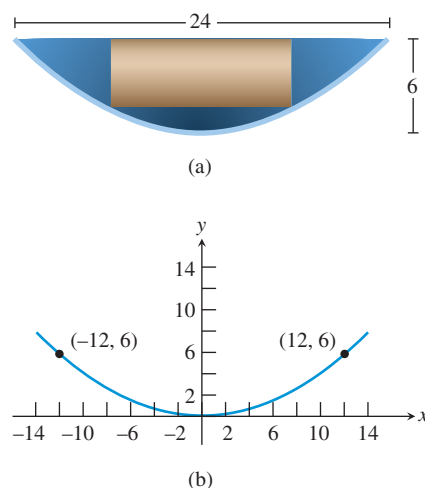


FIGURE 1.82 (a) A parabolic satellite dish with a protective cardboard cylinder in the middle for packaging. (b) The parabola in the coordinate plane. (Example 3)

$$y = kx^2 \quad \text{Vertical shrink}$$

$$6 = k(\pm 12)^2 \quad \text{Substitute } x = \pm 12, y = 6.$$

$$k = \frac{6}{144} = \frac{1}{24} \quad \text{Solve for } k.$$

$$\text{Thus, } y = \frac{1}{24}x^2.$$

To find the height of the cardboard cylinder, we first find the y -coordinate of the parabola 6 inches from the center, that is, when $x = 6$:

$$y = \frac{1}{24}(6)^2 = 1.5$$

From that point to the top of the dish is $6 - 1.5 = 4.5$ in.

Now try Exercise 35.

Although Example 3 serves nicely as a “functions from graphs” example, it is also an example of a function that must be constructed by gathering relevant information from a verbal description and putting it together in the right way. People who do mathematics for a living are accustomed to confronting that challenge regularly as a necessary first step in modeling the real world. In honor of its importance, we have saved it until last to close out this chapter in style.

Functions from Verbal Descriptions

There is no fail-safe way to form a function from a verbal description. It can be hard work, frequently a good deal harder than the mathematics required to solve the problem once the function has been found. The 4-step problem-solving process in Section 1.1 gives you several valuable tips, perhaps the most important of which is to *read* the problem carefully. Understanding what the words say is critical if you hope to model the situation they describe.



EXAMPLE 4 Finding the Model and Solving

Grain is leaking through a hole in a storage bin at a constant rate of 8 cubic inches per minute. The grain forms a cone-shaped pile on the ground below. As it grows, the height of the cone always remains equal to its radius. If the cone is one foot tall now, how tall will it be in one hour?

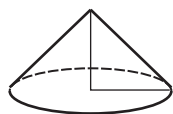


FIGURE 1.83 A cone with equal height and radius. (Example 4)

SOLUTION

Reading the problem carefully, we realize that the formula for the volume of the cone is needed (Figure 1.83). From memory or by looking it up, we get the formula $V = (1/3)\pi r^2 h$. A careful reading also reveals that the height and the radius are always equal, so we can get volume directly as a function of height: $V = (1/3)\pi h^3$.

When $h = 12$ in., the volume is $V = (\pi/3)(12)^3 = 576\pi$ in.³.

One hour later, the volume will have grown by $(60 \text{ min})(8 \text{ in.}^3/\text{min}) = 480 \text{ in.}^3$. The total volume of the pile at that point will be $(576\pi + 480) \text{ in.}^3$. Finally, we use the volume formula once again to solve for h :

$$\begin{aligned}\frac{1}{3}\pi h^3 &= 576\pi + 480 \\ h^3 &= \frac{3(576\pi + 480)}{\pi} \\ h &= \sqrt[3]{\frac{3(576\pi + 480)}{\pi}}\end{aligned}$$

$$h \approx 12.98 \text{ inches.}$$

Now try Exercise 37.

EXAMPLE 5 Letting Units Work for You

How many rotations does a 15 in. (radius) tire make per second on a sport utility vehicle traveling 70 mph?

SOLUTION It is the perimeter of the tire that comes in contact with the road, so we first find the circumference of the tire:

$$C = 2\pi r = 2\pi(15) = 30\pi \text{ in.}$$

That means that 1 rotation = 30π in. From this point we proceed by converting “miles per hour” to “rotations per second” by a series of **conversion factors** that are really factors of 1:

$$\begin{aligned} \frac{70 \text{ miles}}{1 \text{ hour}} &\times \frac{1 \text{ hour}}{60 \text{ min}} \times \frac{1 \text{ min}}{60 \text{ sec}} \times \frac{5280 \text{ feet}}{1 \text{ mile}} \times \frac{12 \text{ inches}}{1 \text{ foot}} \times \frac{1 \text{ rotation}}{30\pi \text{ inches}} \\ &= \frac{70 \times 5280 \times 12 \text{ rotations}}{60 \times 60 \times 30\pi \text{ sec}} \approx 13.07 \text{ rotations per second} \end{aligned}$$

Now try Exercise 39.

Functions from Data

In this course we will use the following 3-step strategy to construct functions from data.

Constructing a Function from Data

Given a set of data points of the form (x, y) , to construct a formula that approximates y as a function of x :

1. Make a scatter plot of the data points. The points do not need to pass the vertical line test.
2. Determine from the shape of the plot whether the points seem to follow the graph of a familiar type of function (line, parabola, cubic, sine curve, etc.).
3. Transform a basic function of that type to fit the points as closely as possible.

Step 3 might seem like a lot of work, and for earlier generations it certainly was; it required all of the tricks of Section 1.6 and then some. We, however, will gratefully use technology to do this “curve-fitting” step for us, as shown in Example 6.

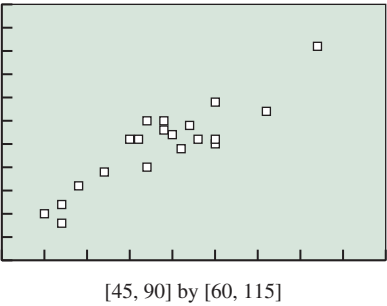


FIGURE 1.84 The scatter plot of the temperature data in Example 6.

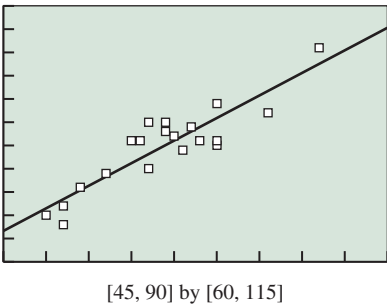


FIGURE 1.85 The temperature scatter plot with the regression line shown. (Example 6)

EXAMPLE 6 Curve-Fitting with Technology

Table 1.12 records the low and high daily temperatures observed on 9/9/1999 in 20 major American cities. Find a function that approximates the high temperature (y) as a function of the low temperature (x). Use this function to predict the high temperature that day for Madison, WI, given that the low was 46.

SOLUTION The scatter plot is shown in Figure 1.84.



Table 1.12 Temperature on 9/9/99

City	Low	High	City	Low	High
New York, NY	70	86	Miami, FL	76	92
Los Angeles, CA	62	80	Honolulu, HI	70	85
Chicago, IL	52	72	Seattle, WA	50	70
Houston, TX	70	94	Jacksonville, FL	67	89
Philadelphia, PA	68	86	Baltimore, MD	64	88
Albuquerque, NM	61	86	St. Louis, MO	57	79
Phoenix, AZ	82	106	El Paso, TX	62	90
Atlanta, GA	64	90	Memphis, TN	60	86
Dallas, TX	65	87	Milwaukee, WI	52	68
Detroit, MI	54	76	Wilmington, DE	66	84

Source: AccuWeather, Inc.

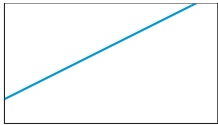
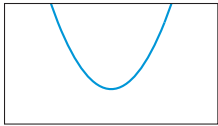
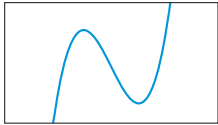
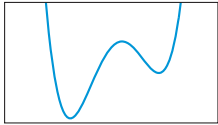
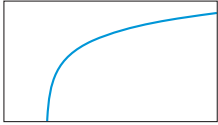
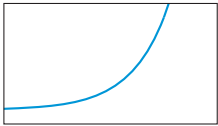
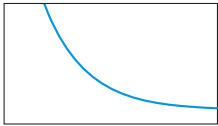
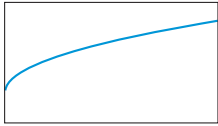
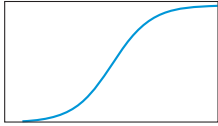
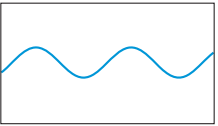
Notice that the points do not fall neatly along a well-known curve, but they do seem to fall *near* an upwardly sloping line. We therefore choose to model the data with a function whose graph is a line. We could fit the line by sight (as we did in Example 5 in Section 1.1), but this time we will use the calculator to find the line of “best fit,” called the **regression line**. (See your grapher’s owner’s manual for how to do this.) The regression line is found to be approximately $y = 0.97x + 23$. As Figure 1.85 shows, the line fits the data as well as can be expected.

If we use this function to predict the high temperature for the day in Madison, WI, we get $y = 0.97(46) + 23 = 67.62$. (For the record, the high that day was 67.)

Now try Exercise 47, parts (a) and (b).

Professional statisticians would be quick to point out that this function should not be trusted as a model for all cities, despite the fairly successful prediction for Madison. (For example, the prediction for San Francisco, with a low of 54 and a high of 64, is off by more than 11 degrees.) *The effectiveness of a data-based model is highly dependent on the number of data points and on the way they were selected.* The functions we construct from data in this book should be analyzed for how well they model the data, not for how well they model the larger population from which the data came.

In addition to lines, we can model scatter plots with several other curves by choosing the appropriate regression option on a calculator or computer. The options to which we will refer in this book (and the chapters in which we will study them) are shown in the following table:

Regression Type	Equation	Graph	Applications
Linear (Chapter 2)	$y = ax + b$		Fixed cost plus variable cost, linear growth, free-fall velocity, simple interest, linear depreciation, many others
Quadratic (Chapter 2)	$y = ax^2 + bx + c$ (requires at least 3 points)		Position during free fall, projectile motion, parabolic reflectors, area as a function of linear dimension, quadratic growth, etc.
Cubic (Chapter 2)	$y = ax^3 + bx^2 + cx + d$ (requires at least 4 points)		Volume as a function of linear dimension, cubic growth, miscellaneous applications where quadratic regression does not give a good fit
Quartic (Chapter 2)	$y = ax^4 + bx^3 + cx^2 + dx + e$ (requires at least 5 points)		Quartic growth, miscellaneous applications where quadratic and cubic regression do not give a good fit
Natural logarithmic (ln) (Chapter 3)	$y = a + b \ln x$ (requires $x > 0$)		Logarithmic growth, decibels (sound), Richter scale (earthquakes), inverse exponential models
Exponential ($b > 1$) (Chapter 3)	$y = a \cdot b^x$		Exponential growth, compound interest, population models
Exponential ($0 < b < 1$) (Chapter 3)	$y = a \cdot b^x$		Exponential decay, depreciation, temperature loss of a cooling body, etc.
Power (requires $x, y > 0$) (Chapter 2)	$y = a \cdot x^b$		Inverse-square laws, Kepler's third law
Logistic (Chapter 3)	$y = \frac{c}{1 + a \cdot e^{-bx}}$		Logistic growth: spread of a rumor, population models
Sinusoidal (Chapter 4)	$y = a \sin (bx + c) + d$		Periodic behavior: harmonic motion, waves, circular motion, etc.

DISPLAYING DIAGNOSTICS

If your calculator is giving regression formulas but not displaying the values of r or r^2 or R^2 , you may be able to fix that. Go to the CATALOG menu and choose a command called “DiagnosticOn.” Enter the command on the home screen and see the reply “Done.” Your next regression should display the diagnostic values.

These graphs are only examples, as they can vary in shape and orientation. (For example, any of the curves could appear upside-down.) The grapher uses various strategies to fit these curves to the data, most of them based on combining function composition with linear regression. Depending on the regression type, the grapher may display a number r called the **correlation coefficient** or a number r^2 or R^2 called the **coefficient of determination**. In either case, a useful “rule of thumb” is: *the closer the absolute value of this number is to 1, the better the curve fits the data.*

We can use this fact to help choose a regression type, as in Exploration 1.

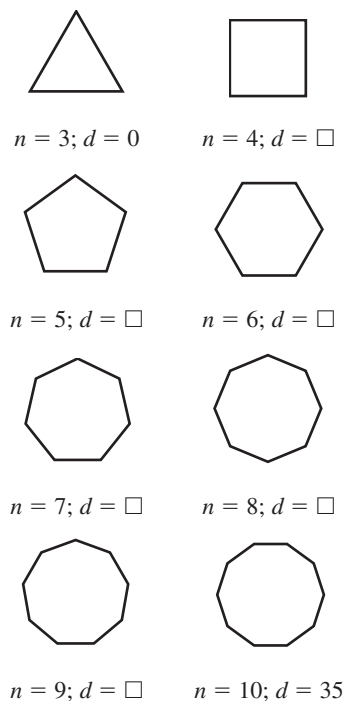


FIGURE 1.86 Some Polygons.
(Exploration 1)

EXPLORATION 1 Diagonals of a Regular Polygon

How many diagonals does a regular polygon have? Can the number be expressed as a function of the number of sides? Try this Exploration.

1. Draw in all the diagonals (i.e., segments connecting nonadjacent points) in each of the regular polygons shown and fill in the number (d) of diagonals in the space below the figure. The values of d for the triangle ($n = 3$) and the decagon ($n = 10$) are filled in for you.
2. Put the values of n in list L1, *beginning with $n = 4$* . (We want to avoid that $y = 0$ value for some of our regressions later.) Put the corresponding values of d in list L2. Display a scatter plot of the ordered pairs.
3. The graph shows an increasing function with some curvature, but it is not clear which kind of growth would fit it best. Try these regressions (preferably in the given order) and record the value of r^2 or R^2 for each: linear, power, quadratic, cubic, quartic. (Note that the curvature is not right for logarithmic, logistic, or sinusoidal curve-fitting, so it is not worth it to try those.)
4. What kind of curve is the best fit? (It might appear at first that there is a tie, but look more closely at the functions you get.) How good is the fit?
5. Looking back, could you have predicted the results of the cubic and quartic regressions after seeing the result of the quadratic regression?
6. The “best-fit” curve gives the actual formula for d as a function of n . (In Chapter 9 you will learn how to derive this formula for yourself.) Use the formula to find the number of diagonals of a 128-gon.

We will have more to say about curve fitting as we study the various function types in later chapters.

CHAPTER OPENER PROBLEM (from page 69)

PROBLEM: The table below shows the growth in the computer price index (CPI) for housing for selected years between 1980 and 2003 (based on 1983 dollars). How can we construct a function to predict the housing CPI for the years 2004–2010?

Computer Price Index (Housing)

Year	Housing CPI
1980	81.1
1985	107.7
1990	128.5
1995	148.5
1998	160.4
1999	163.9
2000	169.6
2001	176.4
2002	180.3
2003	184.8

Source: Bureau of Labor Statistics, quoted in *The World Almanac and Book of Facts 2005*.

SOLUTION: A scatter plot of the data is shown in Figure 1.87, where x is the number of years since 1980. Since the data points fall near an upward sloping line, we can use a calculator to compute a regression line to model the data. The equation of the regression line is found to be $y = 4.37x + 83.20$.

As Figure 1.88 shows, the line fits the data very well.

To predict the housing CPI for 2004, use $x = 24$ in the equation of the regression line. Similarly, we can predict the housing CPI for each of the years 2004–2010, as shown below:

Predicted CPI (Housing)

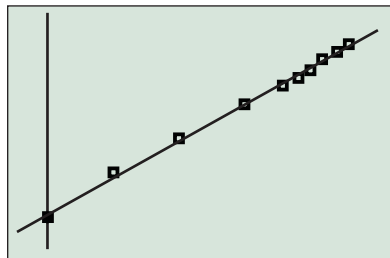
Year	Predicted Housing CPI
2004	$y = 4.37(24) + 83.20 = 188.1$
2005	$y = 4.37(25) + 83.20 = 192.5$
2006	$y = 4.37(26) + 83.20 = 196.8$
2007	$y = 4.37(27) + 83.20 = 201.2$
2008	$y = 4.37(28) + 83.20 = 205.6$
2009	$y = 4.37(29) + 83.20 = 209.9$
2010	$y = 4.37(30) + 83.20 = 214.3$

Even with a regression fit as impressive as in Figure 1.88, it is risky to predict even this far beyond the data set. Statistics like the CPI are dependent on many volatile factors that can quickly render any mathematical model obsolete. In fact, many economists, convinced that this growth could not be sustained, began warning in 2003 that the “housing bubble” would pop before 2010.



$[-2, 25]$ by $[63, 202]$

FIGURE 1.87 Scatter plot of the data for the housing Consumer Price Index.



$[-2, 25]$ by $[63, 202]$

FIGURE 1.88 Scatter plot with the regression line shown.

QUICK REVIEW 1.7 (For help, go to Section P.3 and P.4.)

In Exercises 1–10, solve the given formula for the given variable.

1. **Area of a Triangle** Solve for h : $A = \frac{1}{2}bh$

2. **Area of a Trapezoid** Solve for h : $A = \frac{1}{2}(b_1 + b_2)h$

3. **Volume of a Right Circular Cylinder** Solve for h :
 $V = \pi r^2 h$

4. **Volume of a Right Circular Cone** Solve for h :
 $V = \frac{1}{3}\pi r^2 h$

5. **Volume of a Sphere** Solve for r : $V = \frac{4}{3}\pi r^3$

6. **Surface Area of a Sphere** Solve for r : $A = 4\pi r^2$

7. **Surface Area of a Right Circular Cylinder** Solve for h :
 $A = 2\pi rh + 2\pi r^2$

8. **Simple Interest** Solve for t : $I = Prt$

9. **Compound Interest** Solve for P : $A = P\left(1 + \frac{r}{n}\right)^{nt}$

10. **Free-Fall from Height H** Solve for t : $s = H - \frac{1}{2}gt^2$

SECTION 1.7 EXERCISES

In Exercises 1–10, write a mathematical expression for the quantity described verbally:

- Five more than three times a number x .
- A number x increased by 5 and then tripled.
- Seventeen percent of a number x .
- Four more than 5% of a number x .
- Area of a Rectangle** The area of a rectangle whose length is 12 more than its width x .
- Area of a Triangle** The area of a triangle whose altitude is 2 more than its base length x .
- Salary Increase** A salary after a 4.5% increase, if the original salary is x dollars.
- Income Loss** Income after a 3% drop in the current income of x dollars.
- Sale Price** Sale price of an item marked x dollars, if 40% is discounted from the marked price.
- Including Tax** Actual cost of an item selling for x dollars if the sales tax rate is 8.75%.

In Exercises 11–14, choose a variable and write a mathematical expression for the quantity described verbally.

- Total Cost** The total cost is \$34,500 plus \$5.75 for each item produced.
- Total Cost** The total cost is \$28,000 increased by 9% plus \$19.85 for each item produced.
- Revenue** The revenue when each item sells for \$3.75.
- Profit** The profit consists of a franchise fee of \$200,000 plus 12% of all sales.

In Exercises 15–20, write the specified quantity as a function of the specified variable. It will help in each case to draw a picture.

- The height of a right circular cylinder equals its diameter. Write the volume of the cylinder as a function of its radius.
- One leg of a right triangle is twice as long as the other. Write the length of the hypotenuse as a function of the length of the shorter leg.
- The base of an isosceles triangle is half as long as the two equal sides. Write the area of the triangle as a function of the length of the base.
- A square is inscribed in a circle. Write the area of the square as a function of the radius of the circle.
- A sphere is contained in a cube, tangent to all six faces. Find the surface area of the cube as a function of the radius of the sphere.
- An isosceles triangle has its base along the x -axis with one base vertex at the origin and its vertex in the first quadrant on the graph of $y = 6 - x^2$. Write the area of the triangle as a function of the length of the base.

In Exercises 21–36, write an equation for the problem and solve the problem.

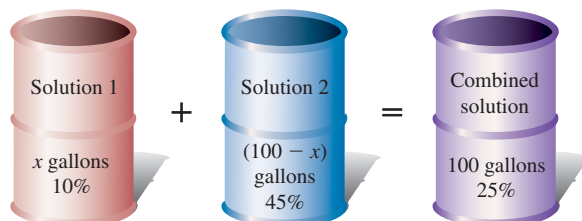
- One positive number is 4 times another positive number. The sum of the two numbers is 620. Find the two numbers.
- When a number is added to its double and its triple, the sum is 714. Find the three numbers.
- Salary Increase** Mark received a 3.5% salary increase. His salary after the raise was \$36,432. What was his salary before the raise?



- 24. Consumer Price Index** The consumer price index for food and beverages in 2003 was 184.0 after a hefty 2.3% increase from the previous year. What was the consumer price index for food and beverages in 2002? (Source: *U.S. Bureau of Labor Statistics*)
- 25. Travel Time** A traveler averaged 52 miles per hour on a 182-mile trip. How many hours were spent on the trip?
- 26. Travel Time** On their 560-mile trip, the Bruins basketball team spent two more hours on the interstate highway than they did on local highways. They averaged 45 mph on local highways and 55 mph on the interstate highways. How many hours did they spend driving on local highways?
- 27. Sale Prices** At a shirt sale, Jackson sees two shirts that he likes equally well. Which is the better bargain, and why?



- 28. Job Offers** Ruth is weighing two job offers from the sales departments of two competing companies. One offers a base salary of \$25,000 plus 5% of gross sales; the other offers a base salary of \$20,000 plus 7% of gross sales. What would Ruth's gross sales total need to be to make the second job offer more attractive than the first?
- 29. Personal Computers** From 1996 to 1997, the worldwide shipments of personal computers grew from 71,065,000 to 82,400,000. What was the percentage increase in worldwide personal computer shipments? (Source: *Dataquest*)
- 30. Personal Computers** From 1996 to 1997, the U.S. shipments of personal computers grew from 26,650,000 to 30,989,000. What was the percentage increase in U.S. personal computer shipments? (Source: *Dataquest*)
- 31. Mixing Solutions** How much 10% solution and how much 45% solution should be mixed together to make 100 gallons of 25% solution?



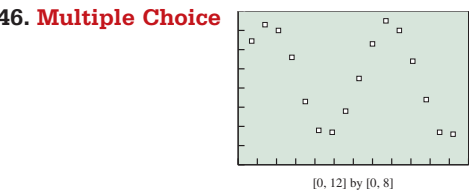
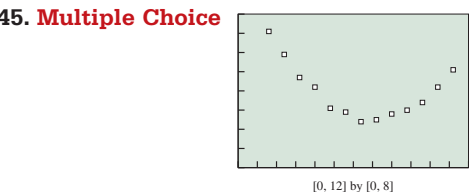
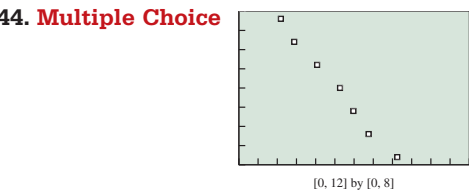
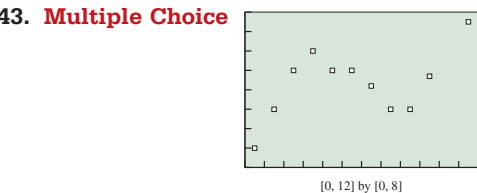
- (a) Write an equation that models this problem.
(b) Solve the equation graphically.
- 32. Mixing Solutions** The chemistry lab at the University of Hardwoods keeps two acid solutions on hand. One is 20% acid and the other is 35% acid. How much 20% acid solution and how much 35% acid solution should be used to prepare 25 liters of a 26% acid solution?
- 33. Maximum Value Problem** A square of side x inches is cut out of each corner of a 10 in. by 18 in. piece of cardboard and the sides are folded up to form an open-topped box.
(a) Write the volume V of the box as a function of x .
(b) Find the domain of your function, taking into account the restrictions that the model imposes in x .
(c) Use your graphing calculator to determine the dimensions of the cut-out squares that will produce the box of maximum volume.
- 34. Residential Construction** DDL Construction is building a rectangular house that is 16 feet longer than it is wide. A rain gutter is to be installed in four sections around the 136-foot perimeter of the house. What lengths should be cut for the four sections?
- 35. Protecting an Antenna** In Example 3, suppose the parabolic dish has a 32 in. diameter and is 8 in. deep, and the radius of the cardboard cylinder is 8 in. Now how tall must the cylinder be to fit in the middle of the dish and be flush with the top of the dish?
- 36. Interior Design** Renée's Decorating Service recommends putting a border around the top of the four walls in a dining room that is 3 feet longer than it is wide. Find the dimensions of the room if the total length of the border is 54 feet.
- 37. Finding the Model and Solving** Water is stored in a conical tank with a faucet at the bottom. The tank has depth 24 inches and radius 9 in., and it is filled to the brim. If the faucet is opened to allow the water to flow at a rate of 5 cubic inches per second, what will the depth of the water be after 2 minutes?
- 38. Investment Returns** Reggie invests \$12,000, part at 7% annual interest and part at 8.5% annual interest. How much is invested at each rate if Reggie's total annual interest is \$900?
- 39. Unit Conversion** A tire of a moving bicycle has radius 16 in. If the tire is making 2 rotations per second, find the bicycle's speed in miles per hour.
- 40. Investment Returns** Jackie invests \$25,000, part at 5.5% annual interest and the balance at 8.3% annual interest. How much is invested at each rate if Jackie receives a 1-year interest payment of \$1571?

Standardized Test Questions

41. **True or False** A correlation coefficient gives an indication of how closely a regression line or curve fits a set of data. Justify your answer.
42. **True or False** Linear regression is useful for modeling the position of an object in free fall. Justify your answer.

In Exercises 43–46, tell which type of regression is likely to give the most accurate model for the scatter plot shown without using a calculator.

- (A) Linear regression
- (B) Quadratic regression
- (C) Cubic regression
- (D) Exponential regression
- (E) Sinusoidal regression



Exploration

47. **Manufacturing** The Buster Green Shoe Company determines that the annual cost C of making x pairs of one type of shoe is \$30 per pair plus \$100,000 in fixed overhead costs. Each pair of shoes that is manufactured is sold wholesale for \$50.

- (a) Find an equation that models the cost of producing x pairs of shoes.
- (b) Find an equation that models the revenue produced from selling x pairs of shoes.
- (c) Find how many pairs of shoes must be made and sold in order to break even.
- (d) Graph the equations in (a) and (b). What is the graphical interpretation of the answer in (c)?

48. **Employee Benefits** John’s company issues employees a contract that identifies salary and the company’s contributions to pension, health insurance premiums, and disability insurance. The company uses the following formulas to calculate these values.

Salary	x (dollars)
Pension	12% of salary
Health Insurance	3% of salary
Disability Insurance	0.4% of salary

If John’s total contract with benefits is worth \$48,814.20, what is his salary?

49. **Manufacturing** Queen, Inc., a tennis racket manufacturer, determines that the annual cost C of making x rackets is \$23 per racket plus \$125,000 in fixed overhead costs. It costs the company \$8 to string a racket.
- (a) Find a function $y_1 = u(x)$ that models the cost of producing x unstrung rackets.
- (b) Find a function $y_2 = s(x)$ that models the cost of producing x strung rackets.



- (c) Find a function $y_3 = R_u(x)$ that models the revenue generated by selling x unstrung rackets.
- (d) Find a function $y_4 = R_s(x)$ that models the revenue generated by selling x rackets.
- (e) Graph y_1 , y_2 , y_3 , and y_4 simultaneously in the window $[0, 10,000]$ by $[0, 500,000]$.
- (f) **Writing to Learn** Write a report to the company recommending how they should manufacture their rackets, strung or unstrung. Assume that you can include the viewing window in (e) as a graph in the report, and use it to support your recommendation.

- 50. Hourly Earnings of U.S. Production Workers** The average hourly earnings of U.S. production workers for 1990–2003 are shown in Table 1.13.



Table 1.13 Average Hourly Earnings

Year	Average Hourly Earnings
1990	10.19
1991	10.50
1992	10.76
1993	11.03
1994	11.32
1995	11.64
1996	12.03
1997	12.49
1998	13.00
1999	13.47
2000	14.00
2001	14.53
2002	14.95
2003	15.35

Source: Bureau of Labor Statistics, U.S. Dept. of Labor, as reported in *The World Almanac and Book of Facts 2005*.

- Produce a scatter plot of the hourly earnings (y) as a function of years since 1990 (x).
- Find the linear regression equation. Round the coefficients to the nearest 0.001.
- Does the value of r suggest that the linear model is appropriate?
- Find the quadratic regression equation. (Round the coefficients to the nearest 0.01.)
- Does the value of R^2 suggest that a quadratic model is appropriate?
- Use both curves to predict the housing CPI for the year 2010. How different are the estimates?
- Writing to Learn** Use the results of parts (a)–(f) to explain why it is risky to predict y -values for x -values that are not very close to the data points, even when the regression plots fit the data points very well.

Extending the Ideas

- 51. Newton's Law of Cooling** A 190° cup of coffee is placed on a desk in a 72° room. According to Newton's Law of Cooling, the temperature T of the coffee after t minutes will be $T = (190 - 72)b^t + 72$, where b is a constant that depends on how easily the cooling substance loses heat. The data in Table 1.14 are from a simulated experiment of gathering temperature readings from a cup of coffee in a 72° room at 20 one-minute intervals:

Table 1.14 Cooling a Cup of Coffee

Time	Temp	Time	Temp
1	184.3	11	140.0
2	178.5	12	136.1
3	173.5	13	133.5
4	168.6	14	130.5
5	164.0	15	127.9
6	159.2	16	125.0
7	155.1	17	122.8
8	151.8	18	119.9
9	147.0	19	117.2
10	143.7	20	115.2

- Make a scatter plot of the data, with the times in list L1 and the temperatures in list L2.
 - Store $L2 - 72$ in list L3. The values in L3 should now be an exponential function ($y = a \times b^x$) of the values in L1.
 - Find the exponential regression equation for L3 as a function of L1. How well does it fit the data?
- 52. Group Activity Newton's Law of Cooling** If you have access to laboratory equipment (such as a CBL or CBR unit for your grapher), gather experimental data such as in Exercise 51 from a cooling cup of coffee. Proceed as follows:
- First, use the temperature probe to record the temperature of the room. It is a good idea to turn off fans and air conditioners that might affect the temperature of the room during the experiment. It should be a constant.
 - Heat the coffee. It need not be boiling, but it should be at least 160° . (It also need not be coffee.)
 - Make a new list consisting of the temperature values minus the room temperature. Make a scatter plot of this list (y) against the time values (x). It should appear to approach the x -axis as an asymptote.
 - Find the equation of the exponential regression curve. How well does it fit the data?
 - What is the equation predicted by Newton's Law of Cooling? (Substitute your initial coffee temperature and the temperature of your room for the 190 and 72 in the equation in Exercise 51.)
 - Group Discussion** What sort of factors would affect the value of b in Newton's Law of Cooling? Discuss your ideas with the group.

Math at Work

When I was a kid, I always liked seeing how things work. I read books to see how things worked and I liked to tinker with things like bicycles, roller skates, or whatever. So it was only natural that I would become a mechanical engineer.

I design methods of controlling turbine speed, so a turbine can be used to generate electric power. The turbine is pushed by steam. A valve is opened to let the steam in, and the speed of the turbine is dictated by how much steam is allowed into the system. Once the turbine is spinning it generates electricity, which runs the pumps, which determine the position of the valves. In this way, the turbine actually controls its own speed.

We use oil pressure to make all of this happen. Oil cannot be compressed, so when we push oil inside a tube, the oil opens the valve, which determines how much steam is pushing the turbine. We need the pumps to push the oil, which controls the opening or closing of the valve. The turbine needs to be able to get to speed in a given amount of time, as well as shut down fast enough in case of emergency, and the opening and closing of the valve controls this speed.

While computers do much of the math these days, we use calculus for stress and motion analysis, as well as vibration analysis, fluid flow, and heat analysis. One of the equations we use is:

$$\gamma_1 + \frac{v_1^2}{2} + gh_1 = \gamma_2 + \frac{v_2^2}{2} + gh_2$$



John Jay

This is an *energy balance equation*. The γ 's represent density of the oil, the v 's represent flow velocity of the oil, the g 's represent the force of gravity, and the h 's represent the height of the oil above the point where it exerts pressure. The subscript 1's indicate these quantities at one point in the system, and the subscript 2's indicate these quantities at another point in the system.

What this equation means is that the total energy at one point in the system must equal the total energy at another point in the system. Thus, we know how much energy is exerted against the valve controls, wherever they are located in the system.

CHAPTER 1 Key Ideas

PROPERTIES, THEOREMS, AND FORMULAS

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PROCEDURES

- Root, Zero, x-intercept 76
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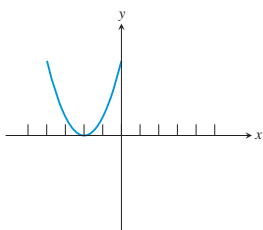
CHAPTER 1 Review Exercises

The collection of exercises marked in **red** could be used as a chapter test.

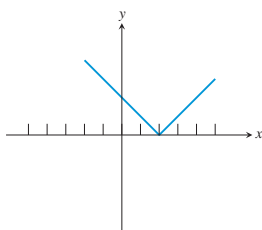
In Exercises 1–10, match the graph with the corresponding function (a)–(j) from the list below. Use your knowledge of function behavior, *not* your grapher.

- | | |
|------------------------------|-------------------------|
| (a) $f(x) = x^2 - 1$ | (b) $f(x) = x^2 + 1$ |
| (c) $f(x) = (x - 2)^2$ | (d) $f(x) = (x + 2)^2$ |
| (e) $f(x) = \frac{x - 1}{2}$ | (f) $f(x) = x - 2 $ |
| (g) $f(x) = x + 2 $ | (h) $f(x) = -\sin x$ |
| (i) $f(x) = e^x - 1$ | (j) $f(x) = 1 + \cos x$ |

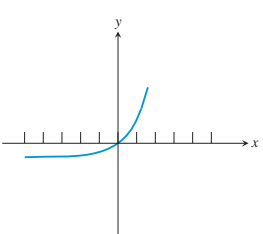
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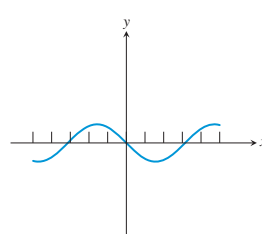
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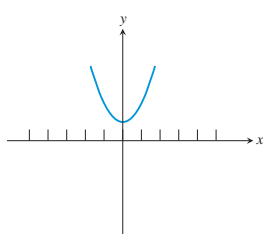
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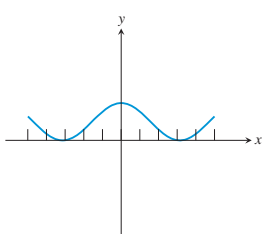
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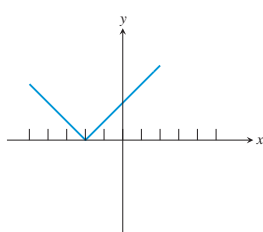
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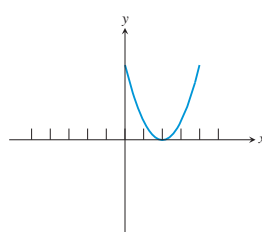
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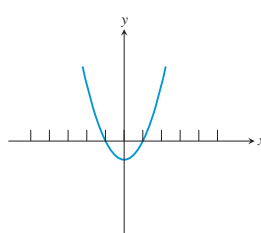
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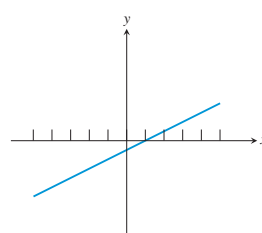
8.



9.



10.



In Exercises 11–18, find (a) the domain and (b) the range of the function.

- | | |
|---------------------------------|---------------------------------------|
| 11. $g(x) = x^3$ | 12. $f(x) = 35x - 602$ |
| 13. $g(x) = x^2 + 2x + 1$ | 14. $h(x) = (x - 2)^2 + 5$ |
| 15. $g(x) = 3 x + 8$ | 16. $k(x) = \sqrt{4 - x^2} - 2$ |
| 17. $f(x) = \frac{x}{x^2 - 2x}$ | 18. $k(x) = \frac{1}{\sqrt{9 - x^2}}$ |

In Exercises 19 and 20, graph the function, and state whether the function is continuous at $x = 0$. If it is discontinuous, state whether the discontinuity is removable or nonremovable.

- | | |
|------------------------------------|---|
| 19. $f(x) = \frac{x^2 - 3}{x + 2}$ | 20. $k(x) = \begin{cases} 2x + 3 & \text{if } x > 0 \\ 3 - x^2 & \text{if } x \leq 0 \end{cases}$ |
|------------------------------------|---|

In Exercises 21–24, find all (a) vertical asymptotes and (b) horizontal asymptotes of the graph of the function. Be sure to state your answers as equations of lines.

- | | |
|--------------------------------------|-----------------------------|
| 21. $y = \frac{5}{x^2 - 5x}$ | 22. $y = \frac{3x}{x - 4}$ |
| 23. $y = \frac{7x}{\sqrt{x^2 + 10}}$ | 24. $y = \frac{ x }{x + 1}$ |

In Exercises 25–28, graph the function and state the intervals on which the function is *increasing*.

- | | |
|-----------------------------|-----------------------------------|
| 25. $y = \frac{x^3}{6}$ | 26. $y = 2 + x - 1 $ |
| 27. $y = \frac{x}{1 - x^2}$ | 28. $y = \frac{x^2 - 1}{x^2 - 4}$ |

In Exercises 29–32, graph the function and tell whether the function is bounded above, bounded below, or bounded.

- | | |
|-------------------------|------------------------------------|
| 29. $f(x) = x + \sin x$ | 30. $g(x) = \frac{6x}{x^2 + 1}$ |
| 31. $h(x) = 5 - e^x$ | 32. $k(x) = 1000 + \frac{x}{1000}$ |

In Exercises 33–36, use a grapher to find all (a) relative maximum values and (b) relative minimum values of the function. Also state the value of x at which each relative extremum occurs.

- | | |
|-----------------------------------|------------------------------|
| 33. $y = (x + 1)^2 - 7$ | 34. $y = x^3 - 3x$ |
| 35. $y = \frac{x^2 + 4}{x^2 - 4}$ | 36. $y = \frac{4x}{x^2 + 4}$ |

In Exercises 37–40, graph the function and state whether the function is odd, even, or neither.

37. $y = 3x^2 - 4|x|$

38. $y = \sin x - x^3$

39. $y = \frac{x}{e^x}$

40. $y = x \cos(x)$

In Exercises 41–44, find a formula for $f^{-1}(x)$.

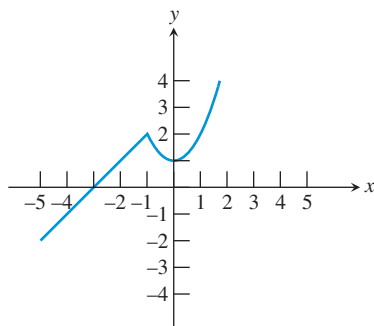
41. $f(x) = 2x + 3$

42. $f(x) = \sqrt[3]{x - 8}$

43. $f(x) = \frac{2}{x}$

44. $f(x) = \frac{6}{x + 4}$

Exercises 45–52 refer to the function $y = f(x)$ whose graph is given below.

45. Sketch the graph of $y = f(x) - 1$.46. Sketch the graph of $y = f(x - 1)$.47. Sketch the graph of $y = f(-x)$.48. Sketch the graph of $y = -f(x)$.

49. Sketch a graph of the inverse relation.

50. Does the inverse relation define y as a function of x ?51. Sketch a graph of $y = f(|x|)$.52. Define f algebraically as a piecewise function. [Hint: the pieces are translations of two of our “basic” functions.]

In Exercises 53–58, let $f(x) = \sqrt{x}$ and let $g(x) = x^2 - 4$.

53. Find an expression for $(f \circ g)(x)$ and give its domain.54. Find an expression for $(g \circ f)(x)$ and give its domain.55. Find an expression for $(fg)(x)$ and give its domain.56. Find an expression for $\left(\frac{f}{g}\right)(x)$ and give its domain.57. Describe the end behavior of the graph of $y = f(x)$.58. Describe the end behavior of the graph of $y = f(g(x))$.

In Exercises 59–64, write the specified quantity as a function of the specified variable. Remember that drawing a picture will help.

59. **Square Inscribed in a Circle** A square of side s is inscribed in a circle. Write the area of the circle as a function of s .60. **Circle Inscribed in a Square** A circle is inscribed in a square of side s . Write the area of the circle as a function of s .61. **Volume of a Cylindrical Tank** A cylindrical tank with diameter 20 feet is partially filled with oil to a depth of h feet. Write the volume of oil in the tank as a function of h .62. **Draining a Cylindrical Tank** A cylindrical tank with diameter 20 feet is filled with oil to a depth of 40 feet. The oil begins draining at a constant rate of 2 cubic feet per second. Write the volume of the oil remaining in the tank t seconds later as a function of t .63. **Draining a Cylindrical Tank** A cylindrical tank with diameter 20 feet is filled with oil to a depth of 40 feet. The oil begins draining at a constant rate of 2 cubic feet per second. Write the depth of the oil remaining in the tank t seconds later as a function of t .64. **Draining a Cylindrical Tank** A cylindrical tank with diameter 20 feet is filled with oil to a depth of 40 feet. The oil begins draining so that the depth of oil in the tank decreases at a constant rate of 2 feet per hour. Write the volume of oil remaining in the tank t hours later as a function of t .65. **U.S. Crude Oil Imports** The imports of crude oil to the U.S. from Canada in the years 1995–2004 (in thousands of barrels per day) are given in Table 1.15.

Table 1.15 Crude Oil Imports from Canada

Year	Barrels/day \times 1000
1995	1,040
1996	1,075
1997	1,198
1998	1,266
1999	1,178
2000	1,348
2001	1,356
2002	1,445
2003	1,549
2004	1,606

Source: Energy Information Administration, Petroleum Supply Monthly, as reported in *The World Almanac and Book of Facts 2005*.

- Sketch a scatter plot of import numbers in the right-hand column (y) as a function of years since 1990 (x).
- Find the equation of the regression line and superimpose it on the scatter plot.
- Based on the regression line, approximately how many thousands of barrels of oil would the U.S. import from Canada in 2010?

66. The winning times in the women's 100-meter freestyle event at the Summer Olympic Games since 1952 are shown in Table 1.16:



Table 1.16 Women's 100-Meter Freestyle

Year	Time	Year	Time
1952	66.8	1980	54.79
1956	62.0	1984	55.92
1960	61.2	1988	54.93
1964	59.5	1992	54.64
1968	60.0	1996	54.50
1972	58.59	2000	53.83
1976	55.65	2004	53.84

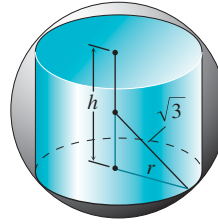
Source: *The World Almanac and Book of Facts 2005*.

- (a) Sketch a scatter plot of the times (y) as a function of the years (x) beyond 1900. (The values of x will run from 52 to 104.)
- (b) Explain why a linear model cannot be appropriate for these times over the long term.
- (c) The points appear to be approaching a horizontal asymptote of $y = 52$. What would this mean about the times in this Olympic event?
- (d) Subtract 52 from all the times so that they will approach an asymptote of $y = 0$. Redo the scatter plot with the new y -values. Now find the *exponential* regression curve and superimpose its graph on the vertically-shifted scatter plot.
- (e) According to the regression curve, what will be the winning time in the women's 100-meter freestyle event at the 2008 Olympics?



67. **Inscribing a Cylinder Inside a Sphere** A right circular cylinder of radius r and height h is inscribed inside a sphere of radius $\sqrt{3}$ inches.

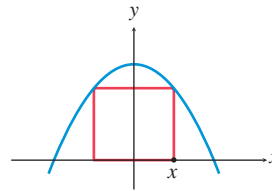
- (a) Use the Pythagorean Theorem to write h as a function of r .



- (b) Write the volume V of the cylinder as a function of r .
- (c) What values of r are in the domain of V ?
- (d) Sketch a graph of $V(r)$ over the domain $[0, \sqrt{3}]$.
- (e) Use your grapher to find the maximum volume that such a cylinder can have.



68. **Inscribing a Rectangle Under a Parabola** A rectangle is inscribed between the x -axis and the parabola $y = 36 - x^2$ with one side along the x -axis, as shown in the figure below.



- (a) Let x denote the x -coordinate of the point highlighted in the figure. Write the area A of the rectangle as a function of x .
- (b) What values of x are in the domain of A ?
- (c) Sketch a graph of $A(x)$ over the domain.
- (d) Use your grapher to find the maximum area that such a rectangle can have.

CHAPTER 1 Project**Modeling the Growth of a Business**

In 1971, Starbucks Coffee opened its first location in Pike Place Market—Seattle’s legendary open-air farmer’s market. By 1987, the number of Starbucks stores had grown to 17 and by 1999 there were 2498 locations. The data in the table below (obtained from Starbucks Coffee’s web site, www.starbucks.com) summarizes the growth of this company from 1987 through 1999.

Year	Number of locations
1987	17
1988	33
1989	55
1990	84
1991	116
1992	165
1993	272
1994	425
1995	676
1996	1015
1997	1412
1998	1886
1999	2498

Explorations

1. Enter the data in the table into your grapher or computer. (Let $t = 0$ represent 1987.) Draw a scatter plot for the data.
2. Refer to page 157 in this chapter. Look at the types of graphs displayed and the associated regression types. Notice that the Exponential Regression model with $b > 1$ seems to most closely match the plotted data. Use your grapher or computer to find an exponential regression equation to model this data set (see your grapher’s guidebook for instructions on how to do this).
3. Use the exponential model you just found to predict the total number of Starbucks locations for 2000 and 2001.
4. There were 2498 Starbucks locations in 2000 and 4709 locations in 2001. (You can verify these numbers and find more up-to-date information in the investors’s section of the Starbucks web site.) Why is there such a big difference between your predicted values and the actual number of Starbucks locations? What real-world feature of business growth was not accounted for in the exponential growth model?
5. You need to model the data set with an equation that takes into account the fact that a business’s growth eventually levels out or reaches a carrying capacity. Refer to page xxx again. Notice that the Logistic Regression modeling graph appears to show exponential growth at first, but eventually levels out. Use your grapher or computer to find the logistic regression equation to model this data set (see your grapher’s guidebook for instructions on how to do this).
6. Use the logistic model you just found to predict the total number of Starbucks locations for 2000 and 2001. How do your predictions compare with the actual number of locations for 2000 and 2001? How many locations do you think there will be in the year 2020?